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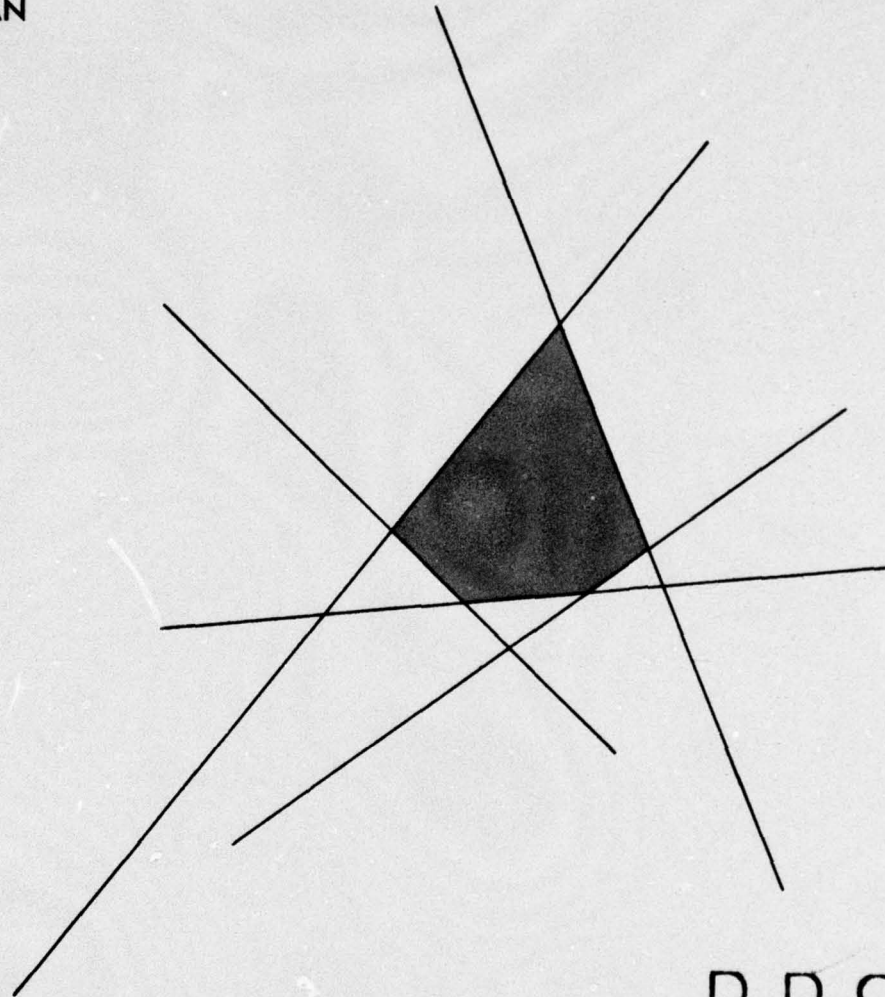
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# GAMBLING MODELS WITH CONCAVE UTILITY FUNCTION

by  
EDUARDO J. SUBELMAN



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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ORC 76-30	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle)  GAMBLING MODELS WITH CONCAVE UTILITY FUNCTION		5. TYPE OF REPORT & PERIOD COVERED  Research Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s)  Eduardo J. Subelman		8. CONTRACT OR GRANT NUMBER(s)  DAAG29-77-G-0040
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS  P-14240-M
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE October 1976
		13. NUMBER OF PAGES 94
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)  Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Gambling Theory Dynamic Programming Stochastic Models Optimal Strategies Adaptive Strategies		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  (SEE ABSTRACT)		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE  
S/N 0102-LF 014-6601

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## GAMBLING MODELS WITH CONCAVE UTILITY FUNCTION

Operations Research Center Research Report No. 76-30

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U. S. Army Research Office - Research Triangle Park

DAAG29-77-G-0040

Operations Research Center  
University of California, Berkeley

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DEDICATION

To my wife, Irene. She knows why.



#### ACKNOWLEDGEMENTS

To my advisor, Professor Sheldon M. Ross, who introduced me to the subject of sequential decision processes, and provided me with constant intellectual stimulus, never failing to lift my spirits when the obstacles seemed insurmountable.

To the other members of the thesis committee, Professor David Gale, and Professor James Pitman for their careful reading.

To the members and staff of the Operations Research Center, University of California, Berkeley, for their support and encouragement.

To Gloria Partee for her skill and patience in typing this thesis.

# ABSTRACT

We examine the problem of a gambler desiring to maximize the expected value of a concave increasing utility function after a fixed, finite number of plays against a casino which offers favourable bets (in the sense of positive expected gains).

The gambles are assumed to have any distribution, which may change from play to play according to a Markov chain, but is known to the player previous to his placing each wager.

Although the optimal wager does not exhibit simple monotonicity properties as a function of the gambler's current fortune, we exhibit two linear functions of the wager and fortune that are indeed monotone and allow to bound above and below the optimal wager for any fortune, given that for any other fortune. These functions generalize the statements "the richer you are the more you save" and "the richer you are the more you should try to get."

The dependence of the optimal bet on the gamble currently available is examined, and, in the case of coin tossing games, it is shown that "the better the gamble, the more you should bet."

The case in which the distribution of the gamble is unknown, except for a prior distribution over a set of possible alternatives is also analyzed and results similar to the above are exhibited.

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## CHAPTER 0

### INTRODUCTION

This thesis is concerned with the study of optimal gambling systems in favourable games with finite horizon, under the assumption that the gambler's utility depends only on his final fortune.

Historically it has been assumed that any gambling situation a player enters must be unfavourable to him. During the past fifteen years however, the existence of favourable or superfair games has been identified by Thorp, Epstein, Kassouf, and Fisher and Lorie among others. Such gambles as blackjack, baccarat, stocks and options have been shown to be superfair. A number of interesting papers on favourable games have been written, and some statistical work has been done in attempting to identify more superfair gambles.

We consider models in which the outcome of each bet is determined by a random variable, whose distribution depends on some "state of the universe." This state of the universe, although known to the gambler, may change from play to play in a random way. The player must choose his wager at each stage of the game, and is assumed to act with the objective of maximizing his utility at the end of the game. We also consider the case where the state of the universe is known to the gambler only in distribution, allowing the gambler to incorporate the information obtained as the outcome of each gamble becomes known.

#### Preliminaries

Probability theory was born out of the efforts of early mathematicians to study gambling problems. Epstein [6] gives an interesting review of the history of probability and its relation to gambling.

The past twenty years have seen a rebirth of interest in gambling theory. Dubins' and Savage's work "How to Gamble if You Must" [5] considers unfavourable gambling situations and the optimal strategies for these games. They prove, among other things that if the gambler's goal is to reach a fortune of  $N$  before going broke, and he is allowed even money wagers with probability  $p < \frac{1}{2}$  of success, then bold play (i.e. bet  $\min \{N - i, i\}$  if your fortune is  $i$ ) is optimal. Ross [16] also considers subfair coin tossing games and shows that playing timidly (i.e. betting the minimum) maximizes the expected playing time until the gambler goes broke.

The first modern day author to consider superfair games is Kelly [11]. He relates gambling to information theory by considering a gambler receiving information on the outcome of an event through a noisy channel, and being able to bet on the event at the original odds. For a binary channel with probability  $p$  of correct transmission, Kelly shows that if the gambler wishes to maximize  $\lim_{n \rightarrow \infty} \frac{1}{n} \ln S_n$  (where  $S_n$  is the gambler's fortune after  $n$  gambles) and he is restricted to betting a constant proportion of his current fortune at each time period, then he should bet the proportion  $(2p - 1)$ . Bellman and Kalaba [1], using Kelly's model, show that betting the proportion  $(2p - 1)$  maximizes  $ELnS_n$  for all  $n$ .

Ross in [16] also considers fair and superfair coin tossing games and shows that timid play maximizes the probability of playing  $n$  games before going broke, for all  $n$  (stochastic maximization). The same result was obtained in the case of fair games, by Freedman [8] and Molenaar and Van Der Velde [13].

Breiman [2] generalizes Kelly's model and proves that for favourable games, the gambling system which maximizes  $E(\ln S_n)$  asymptotically minimizes the expected time until  $S_n \geq x$  (for sufficiently large  $x$ ) and asymptotically maximizes  $P[S_n > y]$  (for sufficiently large  $n$  and  $y$  fixed). Breiman's work is the basis for a number of papers which advocate maximizing  $E \ln S_n$  in the belief that the gambler's objectives are long term.

Ferguson [7] looks at the optimal strategies for coin tossing games in which the gambler's objective is to avoid ruin. He assumes that at each gamble the probability of winning is chosen randomly from a distribution  $F(p)$  and told to the gambler before his placing a bet. Ferguson considers two cases: the gambler must pay \$1.00 after each play or there is a minimum bet of \$1.00. He conjectures that for both models the probability of ruin tends to zero exponentially as the gambler's fortune increases. This fact is proved by Breiman [3]. From this conjecture Ferguson proves that if  $F$  is degenerate at some point  $p_0 > \frac{1}{2}$  then, for his first model, if the gambler is wealthy he should wager approximately 
$$\frac{\ln((1 - p_0)/p_0)}{\ln(4p_0(1 - p_0))}$$
. Truelove [19] extends Ferguson's paper proving that, for the first model, the probability of ruin is continuous in  $x$ , the gambler's fortune.

Thorp [18] gives a good overview of the various games in existence today which are favourable to the gambler.

Epstein [6] has written a book totally devoted to the various gambling situations available, and how to play optimally in them.



Pasternack [14] considers the coin tossing game but discounts money over time and finds conditions under which the gambler is better off by making risk-free investments. He also shows that, when it is optimal to gamble, there exists a proportional betting strategy such that the expected discounted fortune converges to infinity almost surely.

In the theory of consumption and investment Hakansson [9] and Schechtman [17] have considered models in which the gambler must decide at each period which portion of his fortune to consume (and derive a utility from) and which portion to place in a risky investment. It is interesting to note that, although there are apparent similarities between their models and those of a gambler, their results differ from those for the gambler whose utility is paid at the end of his planning horizon. This can be explained by the fact that such gambler lacks an incentive to divert part of his funds into consumption.

This thesis is divided into five chapters and one Appendix. In Chapter I we formulate the gambling model using the dynamic programming recursive equation method and find some general properties of the optimal value function.

Chapter II is devoted to some special utility functions and generalizes the results of Bellman and Kalaba.

Chapter III considers the relation between the gambler's fortune and his betting strategy.

Chapter IV examines the dependence of the betting strategy on the probability of winning in coin tossing games.

Chapter V generalizes these results to the case where the probability of winning is unknown except in distribution.

The Appendix presents results due to Topkis and Veinott which have not been published previously but are fundamental in the development of this work.

Throughout this work the logarithms considered will be natural logarithms and strategy will be taken to mean any gambling system whatsoever, where the wagers made may depend on the history of the game, current fortune, etc. Following Bourbaki, we will use the terms "increasing," "decreasing," "greater than," "less than" to denote respectively "nondecreasing," "nonincreasing," "greater than or equal to," "less than or equal to," and so on, reserving the word "strictly" to mean exactly that. Thus "increasing" means the value does not decrease (but could be constant) and "strictly increasing" means the value does not decrease and is not constant over any interval.

Lemmas, theorems and corollaries are numbered by section in the order in which they appear. The Appendix is referred to as Section A. The end of a proof is signaled by ■, symbol usually attributed to Halmos.

## CHAPTER I

## FORMULATION OF GAMBLING MODEL

Consider the following scenario: A gambler has a fortune of  $x$ . He knows that the state of the mechanism against which he is playing is  $i \in \{0, 1, 2, \dots\}$ . He can place any positive wager  $y$  not exceeding his current fortune ( $0 \leq y \leq x$ ). The mechanism will then be set in motion, and when it stops his fortune will be  $x + Ry$  where  $R$  is a random variable with distribution  $G_i(R)$ , which depends on  $i$ . We will assume  $P[R \geq -1] = 1$ , meaning that the gambler cannot lose more than his wager, and also, that  $\int r dG_i(r) \leq \theta < \infty$ , for all  $i$ .

After this, the state of the gambling mechanism changes to  $j$  with probability  $P_{ij}$ , independent of  $R$ . The gambler is informed of the new state of the mechanism and is allowed to place another wager, the process repeating itself a predetermined number of times,  $n$ .

The gambler's problem is thus to determine his betting strategy, i.e., to define for every conceivable situation he may find himself in, the amount to be wagered.

We will consider a gambler who is interested in maximizing the expected value of his fortune at the end of the  $n$  plays, or, more generally, we will assume that he attaches a utility  $V(x)$  to each possible value of his final fortune and he wishes to maximize his expected utility after  $n$  gambles.

The utility functions we will consider can be characterized by the properties



- a)  $V(x)$  increasing in  $x$
- b)  $V(x)$  concave in  $x$
- c)  $V(x)$  is right continuous at  $x = 0$ .

Assumption a) simply states that the wealthier the gambler is, the happier he is. Clearly if having less money made the gambler happier, he could adopt the strategy of giving away as much as he needed in order to increase his utility. Hence we have no use for utilities that are strictly decreasing over any subset of the positive real line.

Assumption b) characterizes utilities with decreasing marginal returns. It states that if, when your fortune is  $x$ , you need to win  $A_x$  in order to increase your utility by  $\epsilon$ , then, when your fortune is  $x' > x$  you need to win  $A_{x'} > A_x$  in order to increase your utility by the same amount. In simpler terms, if you were poor you would be inclined to accept a menial task to win an extra \$1, but if you were rich, you would probably not accept the same task for only one extra dollar.

Assumption c), as will be seen below, is needed in order to ensure the existence of optimal betting strategies. It does not of course rule out the possibility of unbounded utilities in which  $V(0) = -\infty$ . In fact we will take  $-\infty$  to be (by continuity) the definition of  $\ln 0$  and of  $-0^\beta$   $\beta < 0$ .

Let  $V_n[x, i]$  be the supremum expected utility when there are  $n$  betting opportunities, the gambler's fortune is  $x$  and the mechanism is in state  $i$ . The usual Dynamic Programming formulation leads us to the following functional equations

$$V_n[x, i] = \sup_{0 \leq y \leq x} \left\{ \sum_j P_{ij} \int_{-1}^{\infty} V_{n-1}[x + ry, j] dG_i(r) \right\}$$

$$V_0[x, i] = V[x] .$$

We will write

$$E_{RJ/i} V_{n-1}[x + Ry, J] \equiv \sum_j P_{ij} \int_{-1}^{\infty} V_{n-1}[x + ry, j] dG_i(r)$$

to represent the expectation with respect to  $R$  and  $J$ , whose distributions depend on  $i$ .

We now proceed to show the existence of optimal policies, as well as some smoothness properties of  $V_n[x, i]$ .

Lemma 1.1:

If  $V_{n-1}[x, i]$  is continuous in  $x$  on  $(0, \infty)$  and right continuous at the origin for all  $i$ , then there exists a number  $y_n(x, i)$  such that

$$E_{RJ/i} V_{n-1}[x + Ry_n(x, i), J] = \sup_{0 \leq y \leq x} E_{RJ/i} V_{n-1}[x + Ry, J] .$$

Proof:

$V_{n-1}[x + ry, j]$  is continuous in  $y \in (0, \infty)$  and right continuous at the origin, for all  $r$  and  $j$ . Hence so is  $E_{RJ/i} V_{n-1}[x + Ry, J]$ . Since  $\{y : 0 \leq y \leq x\}$  is compact, the supremum is attained. ■

Recall that  $\int r dG_i(r) \leq \theta < \infty$  for all  $i$ .

Define  $\psi = \max \{1, 1 + \theta\}$ . Also let  $ER_i = \int r dG_i(r)$ .

Theorem 1.2:

For all  $i$ ,  $V_n[x, i]$  is increasing and concave in  $x$ . It is thus continuous in  $x$  on  $(0, \infty)$ . Furthermore,  $V_n[x, i] \leq V[x\psi^n]$  and it is right continuous at the origin. Hence an optimal policy exists.

Proof:

$V_0[x, i] = V[x]$  satisfies all the above by definition. Assume  $V_{n-1}[x, i]$  does. Then the number  $y_n(x, i)$  exists for all  $x$  and  $i$  by Lemma 1.1.

Now, for  $\varepsilon > 0$ , since  $0 \leq y_n(x, i) \leq x + \varepsilon$

$$\begin{aligned} V_n[x + \varepsilon, i] &\geq E_{R, J/i} V_{n-1}[x + \varepsilon + Ry_n(x, i), J] \\ &\geq E_{RJ/i} V_{n-1}[x + Ry_n(x, i), J] = V_n[x, i] \end{aligned}$$

hence  $V_n[x, i]$  is increasing in  $x$ .

For any  $x_1, x_2$  and any  $\lambda \in [0, 1]$ , since  $0 \leq y_n(x_1, i) \leq x_1$  and  $0 \leq y_n(x_2, i) \leq x_2$  imply

$$0 \leq \lambda y_n(x_1, i) + (1 - \lambda) y_n(x_2, i) \leq \lambda x_1 + (1 - \lambda) x_2$$

showing that  $\lambda y_n(x_1, i) + (1 - \lambda) y_n(x_2, i)$  is a feasible wager when our fortune is  $\lambda x_1 + (1 - \lambda) x_2$ , we have



$$V_n[\lambda x_1 + (1 - \lambda)x_2, i] \geq \\ E_{RJ/i} V_{n-1}[\lambda x_1 + (1 - \lambda)x_2 + R(\lambda y_n(x_1, i) + (1 - \lambda)y_n(x_2, i)), J]$$

and by concavity of  $V_{n-1}[x, j]$  in the first argument

$$\geq E_{RJ/i} \{ \lambda V_{n-1}[x_1 + R y_n(x_1, i), J] + (1 - \lambda) V_{n-1}[x_2 + R y_n(x_2, i), J] \}$$

showing

$$V_n[\lambda x_1 + (1 - \lambda)x_2, i] \geq \lambda V_n[x_1, i] + (1 - \lambda) V_n[x_2, i] .$$

Thus  $V_n[x, i]$  is concave in  $x$ , hence continuous in  $x$ , except possibly at the origin.

Since  $V_n[x, i]$  is increasing in  $x$ , clearly  $\lim_{x \rightarrow 0^+} V_n[x, i]$  exists (it may be  $-\infty$ ). It is also evident that  $V_n[0, i] = V[0]$ .

By the concavity of  $V_n[x, j]$  we have as a consequence of Jensen's inequality ([12], p. 159) that for all  $x > 0$  and  $y \in [0, x]$ ,

$$\begin{aligned} \sum_j P_{ij} \int_{-1}^{\infty} V_{n-1}[x + ry, j] dG_i(r) &\leq \sum_j P_{ij} V_{n-1}[x + E R_i y, j] \\ &\leq \sum_j P_{ij} V_{n-1}[x + \theta y, j] . \end{aligned}$$

For  $y \in [0, x]$  we have thus,

$$\sum_j P_{ij} \int_{-1}^{\infty} V_{n-1}[x + ry, j] dG_i(r) \leq \sum_j P_{ij} V_{n-1}[x\psi, j] \leq V[x\psi^n]$$

showing

$$V_n[x, i] \leq V[x\psi^n] \text{ for all } x.$$

Now, if  $V(0) = V_n[0, i] > -\infty$ , then

$$0 \leq V_n[x, i] - V_n[0, i] \leq V[x\psi^n] - V[0]$$

and letting  $x$  go to zero,

$$\lim_{x \rightarrow 0^+} V_n[x, i] = V_n[0, i].$$

If  $V[0] = V_n[0, i] = -\infty$  then

$$V_n[x, i] \leq V[x\psi^n]$$

and letting  $x$  go to zero

$$\lim_{x \rightarrow 0^+} V_n[x, i] = -\infty = V_n[0, i]$$

which proves the right continuity at the origin. ■

It thus follows that the supremum in the optimality equation is always attained and we can and will write it as

$$V_n[x, i] = \max_{0 \leq y \leq x} E_{R, J/i} V_{n-1}[x + Ry, J]$$

$$V_0[x, i] = V[x].$$

We will call  $y_n(x, i)$  the optimal amount to bet as determined from this equation. If it is not unique, we will allow  $y_n(x, i)$  to be either the smallest or the largest of such values, although we

will assume consistency in this choice: always the smallest or always the largest.

Proposition 1.3:

$V_n[x, i]$  is increasing in  $n$ .

Proof:

By deciding to bet zero on the last stage of an  $n$  stage problem, one reduces it to an  $(n - 1)$  stage problem. Thus in the  $n$  stage problem one can do at least as well as in an  $(n - 1)$  stage problem. ■

Proposition 1.4:

Whenever the system is in a state  $i$  such that  $ER_i \leq 0$  then it is optimal to bet zero.

Proof:

$$V_n[x, i] = \max_{0 \leq y \leq x} \sum_j P_{ij} E_{R/i} V_{n-1}[x + Ry, j] .$$

Since, by Theorem 1.2  $V_{n-1}[\cdot, \cdot]$  is concave in the first argument, we may apply Jensen's inequality (cf. [12], p. 159) to obtain

$$V_n[x, i] \leq \max_{0 \leq y \leq x} \sum_j P_{ij} V_{n-1}[x + (ER_i)y, j] .$$

Since, by Theorem 1.2  $V_{n-1}[x, j]$  is increasing in  $x$  for all  $j$  and  $ER_i \leq 0$  we have that



$V_{n-1}[x + (ER_1)y, j]$  is decreasing in  $y$  for all  $j$  hence  
 $\sum P_{ij} V_{n-1}[x + (ER_1)y, j]$  is decreasing in  $y$  and its maximum over  
 $0 \leq y \leq x$  is attained at  $y = 0$ . Thus

$$V_n[x, i] \leq \sum P_{ij} V_{n-1}[x, j] .$$

However, since betting zero is always a feasible strategy we must have

$$V_n[x, i] \geq E_{RJ/i} V_{n-1}[x, J] = \sum P_{ij} V_{n-1}[x, j] .$$

Thus we must have that  $ER_1 \leq 0$  implies

$$V_n[x, i] = \sum P_{ij} V_{n-1}[x, j]$$

showing the optimality of betting zero. ▀

This last result is of course the reason for which concave utility functions are sometimes called "risk averse" (see for example [15]). A gambler whose utility has this property will not bet on a gamble where his expectation is negative. From now on, unless specifically indicated, we will assume  $ER_1 > 0$ , and consider this condition as the definition of a favourable gamble.

So far we have shown that  $V_n[x, i]$  inherits the monotonicity and concavity properties of  $V_0[x, i]$ . Another property which will prove useful later is differentiability.

To this end we establish first

Lemma 1.5:

If  $V_{n-1}[x, i]$  is strictly increasing and differentiable in the first argument at some  $x_0 > 0$  for all  $i$ , then  $y_n(x_0, i) > 0$  for all  $i$ .

Proof:

$$V_n[x_0, i] = \max_{0 \leq y \leq x_0} E_{RJ/i} V_{n-1}[x_0 + Ry, J].$$

Since the function to be maximized is differentiable, we can compute its derivative with respect to  $y$ , evaluated at  $y = 0$  to be

$$\begin{aligned} E_{RJ/i} V'_{n-1}[x_0, J] &= \int_{-1}^{\infty} \sum_j P_{ij} V'_{n-1}[x_0, j] dG_i(r) \\ &= (ER_i) \sum_j P_{ij} V'_{n-1}[x_0, j] > 0 \end{aligned}$$

since  $ER_i > 0$  and  $V'_{n-1}[x_0, j] > 0$  for all  $j$ . Hence

$E_{RJ/i} V_{n-1}[x_0 + Ry, J]$  is strictly increasing in  $y$  at  $y = 0$  and the optimal  $y$  must be strictly positive. ■

We can now prove

Theorem 1.6:

If  $V[x]$  is strictly increasing and differentiable at  $x > 0$  then so is  $V_n[x, i]$  for all  $n$  and all  $i$ .

Proof:

Our proof follows that of Schechtman's theorem 2.2 [17].  $V_0[x, i] \equiv V[x]$  is strictly increasing and differentiable at  $x > 0$  for all  $i$ .

Assume  $V_{n-1}[x, i]$  has this property at  $x_0$ . By Theorem 1.2  $V_n[x, i]$  is concave. Thus it will be differentiable at  $x_0$  if and only if it has a unique support at this point.

Let  $\bar{y} = y_n(x_0, i)$ ,  $\bar{u} = x_0 - \bar{y}$ . By Lemma 1.5  $\bar{y} > 0$ .

Let  $S_0$  be a support of  $V_n[x, i]$  at  $x_0$ . By concavity of  $V_n[x, i]$  we must have that, for any  $x \geq 0$

$$V_n[x, i] - S_0 x \leq V_n[x_0, i] - S_0 x_0.$$

For any  $y \geq 0$ ,  $u \geq 0$ , setting  $x = y + u$  we also must have, since  $y$  is not necessarily the optimal bet at  $x$

$$E_{RJ/i} V_{n-1}[x + Ry, J] \leq V_n[x, i].$$

Substituting above we obtain

$$E_{RJ/i} V_{n-1}[u + (1+R)y, J] - S_0 u - S_0 y \leq E_{RJ/i} V_{n-1}[\bar{u} + (1+R)\bar{y}, J] - S_0 \bar{u} - S_0 \bar{y}.$$

Since  $u$  was arbitrary, we can set it equal to  $\bar{u}$  obtaining that for all  $y \geq 0$

$$E_{RJ/i} V_{n-1}[\bar{u} + (1+R)y, J] - S_0 y \leq E_{RJ/i} V_{n-1}[\bar{u} + (1+R)\bar{y}, J] - S_0 \bar{y}.$$

Thus we see that the differentiable concave function of  $y$ ,  $E_{RJ/i} V_{n-1}[\bar{u} + (1+R)y, J] - S_0 y$  attains its maximum over  $y \geq 0$  at  $\bar{y} > 0$ . This implies that its derivative with respect to  $y$  vanishes at  $\bar{y}$ , hence

$$E_{RJ/i} (1+R) V'_{n-1}[\bar{u} + (1+R)\bar{y}, J] - S_0 = 0$$



or

$$S_0 = E_{RJ/i} (1 + R) V'_{n-1} [x_0 + R\bar{y}, J] .$$

Thus the support  $S_0$  must be unique, showing the differentiability of  $V_n[x, i]$ . We also obtain a useful recursive equation for the derivatives of  $V_n[x, i]$ , namely,

$$V'_n[x, i] = E_{RJ/i} (1 + R) V'_{n-1} [x + Ry, J] .$$

Also, since  $V'_{n-1} [x + R\bar{y}, J] > 0$  and  $(1 + R) \geq 0$ , we see that  $V'_n[x, i] > 0$  (unless  $P[R = -1] = 1$  which is ruled out by the fact that  $ER_1 > 0$ ) showing that  $V_n[x, i]$  is strictly increasing in  $x$ . ■

We now turn to the examination of the dependence of  $V_n[x, i]$  on its second argument. Clearly not much can be said unless one imposes some structure on the random variables that determine the outcome of the gamble. The type of result we are interested in is, in broad terms, "the better the gamble available to you, the better off you are." To this effect we will assume for the remainder of this section that the random variables  $R_i$  are stochastically increasing in  $i$ , or, in mathematical terms  $P[R_i \geq r]$  is increasing in  $i$  for all  $r$ .

As is well known, if  $\phi$  is an increasing function then the stochastic ordering of the  $R_i$  implies  $E\phi(R_i)$  is increasing in  $i$ .

The application of stochastic monotonicity to markov chains yields the concept of monotone matrices (Keilson and Kester [10], Daley [4]). The transition matrix of a markov chain,  $P = [P_{ij}]$

is said to be monotone if and only if  $\sum_{j=k}^{\infty} P_{ij}$  is increasing in  $i$  for all  $k$ . This merely says that the rows of the matrix correspond to the distributions of a stochastically increasing sequence of random variables or, in simpler terms, the higher the state of the system, the stochastically higher the state it will jump to next.

It thus follows that if  $f_j$  is an increasing sequence and if  $P$  is monotone, then  $\sum_j f_j P_{ij}$  is increasing in  $i$ .

We are now in a position to prove

Proposition 1.7:

Assume the  $R_i$  are stochastically increasing in  $i$  and the matrix of transition probabilities is monotone. Then  $V_n[x, i]$  is increasing in  $i$ .

Proof:

$V_0[x, i] = V[x]$  is increasing in  $i$ .

Assume  $V_{n-1}[x, i]$  is increasing in  $i$ .

Then it follows that

$$\int V_{n-1}[x + ry, j] dG_i(r)$$

is increasing in  $j$ . If  $y \geq 0$  then, by Theorem 1.2  $V_{n-1}[x + ry, j]$  is increasing in  $r$ . Therefore, by the stochastic monotonicity of  $R_i$

$$\int V_{n-1}[x + ry, j] dG_i(r)$$

is increasing in  $i$ , as well as in  $j$ .

By the monotone property of the transition matrix

$$\sum_j P_{ij} \int V_{n-1}[x + ry, j] dG_i(r)$$

is increasing in  $i$  for all  $y \geq 0$ . Thus

$$V_n[x, i] = \max_{0 \leq y \leq x} \sum_j P_{ij} \int V_{n-1}[x + ry, j] dG_i(r)$$

is increasing in  $i$ . ■

As a special case of the above proposition we see that if the state of the gambling system does not change from play to play, then for stochastically greater gambles we obtain greater expected utilities. This follows from the fact that the transition matrix in this case is the identity matrix which is monotone.

It is a trivial task to construct counter-examples to show that the result no longer holds if one merely assumes that the gambles are ordered stochastically and does not impose special conditions on the transition matrix.

Let the utility function be  $V[x] = x$ . As a consequence of Theorem 2.2 we will see that in this case one should always bet his current fortune (if the gamble is favourable) i.e.  $y_n(x, i) = x \quad \forall n$ ,  $\forall i \in ER_1 > 0$ .

The three gambles available are coin tosses with probabilities of winning .6, .7 and .8 respectively. Thus, clearly, the corresponding random variables are stochastically ordered.

The transition matrix will be taken as



$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which is clearly not monotone.

It is easy to verify then that

$$V_n[x,2] = (.7)^n 2^n x$$

$$V_n[x,3] = .8(.6)^{n-1} 2^n x$$

and obviously  $V_n[x,3] < V_n[x,2]$  for all  $n \geq 2$  which is true because after playing the best bet once, one is forced to continue with the worst one from then on, whereas, starting with the second best bet one will always face that one.

## CHAPTER II

### SOME SPECIAL UTILITY FUNCTIONS

As a preamble to the analysis of the betting strategy in the general case, we consider some special forms of utility functions which seem to be fairly popular among economists. We first consider  $V[x] = \text{Ln} x$  which was essentially the utility analyzed by Kelly.

#### Theorem 2.1:

If  $V[x] = \text{Ln} x$  then  $V_n[x, i] = V_n[1, i] + \text{Ln} x$  and the optimal wager is given by  $y_n(x, i) = \alpha(i)x$  where  $\alpha(i)$  does not depend on  $n$  or  $x$ .

#### Proof:

Instead of choosing, at each stage, the amount to bet the gambler could equivalently take as his decision variable the fraction of his current fortune to be wagered. The dynamic programming recursive equations now become

$$V_n[x, i] = \max_{0 \leq \alpha \leq 1} E_{RJ/i} V_{n-1}[x(1 + \alpha R), J] .$$

The proof then proceeds by induction.

$V_0[x, i] = \text{Ln} x$  which is clearly of the required form since  $V_0[1, i] = \text{Ln} 1 = 0$ .

Assume  $V_{n-1}[x, i] = V_{n-1}[1, i] + \text{Ln} x$ .

$$\begin{aligned} V_n[x, i] &= \max_{\alpha \in [0, 1]} \{E_{RJ/i} V_{n-1}[1, J] + E_{RJ/i} \text{Ln}[x(1 + \alpha R)]\} \\ &= E_{RJ/i} V_{n-1}[1, J] + \text{Ln} x + \max_{\alpha \in [0, 1]} E_{RJ/i} \text{Ln}(1 + \alpha R) . \end{aligned}$$

By setting  $x = 1$  we have, since  $\ln 1 = 0$ , that

$$V_n[1,i] = E_{R/J/i} V_{n-1}[1,J] + \max_{\alpha \in [0,1]} E_{R/J/i} \ln(1 + \alpha R)$$

which, compared to the equation preceeding it shows that

$$V_n[x,i] = V_n[1,i] + \ln x$$

thus completing the induction. Note also that the value of the optimal  $\alpha$  is determined by

$$\max_{\alpha \in [0,1]} E_{R/J/i} \ln(1 + \alpha R)$$

which shows that the fraction of the current fortune to be wagered is independent of  $n$  and  $x$ . ■

This result generalizes that of Bellman and Kalaba [1] for coin tossing games. It is interesting to note that the policy is absolutely myopic, that is, it does not consider the planning horizon or the possibility of the appearance of better (or worse) gambles in the future.

Another interesting utility is  $x^\beta/\beta$  for a constant  $\beta \leq 1$ ,  $\beta \neq 0$ . It is obvious that the utility considered above is a limiting case of this one, as  $\beta$  goes to 0. A similar result to the one above holds for this utility.

#### Theorem 2.2:

If  $V[x] = x^\beta/\beta$   $\beta \leq 1$   $\beta \neq 0$  then  $V_n[x,i] = V_n[1,i]x^\beta$  and the optimal wager is given by  $y_n(x,i) = \alpha(i)x$  where  $\alpha(i)$  does not depend on  $n$  or  $x$ .



Proof:

We will only consider the case  $\beta > 0$ . The proof in the case  $\beta < 0$  is identical except that one must consider the fact that the utility is negative.

Formulating the problem in terms of the fraction of the current fortune to be wagered yields as before

$$V_n[x, i] = \max_{\alpha \in [0, 1]} E_{RJ/i} V_{n-1}[x(1 + \alpha R), J]$$

$V_0[x, i] \equiv \frac{1}{\beta} x^\beta$  is of the required form since  $1^\beta = 1$ .

Assume  $V_{n-1}[x, i] = V_{n-1}[1, i] x^\beta$

$$\begin{aligned} V_n[x, i] &= \max_{\alpha \in [0, 1]} \left\{ \sum_j P_{ij} \int_{-1}^{\infty} V_{n-1}[1, j] [x(1 + \alpha r)]^\beta dG_i(r) \right\} \\ &= x^\beta \left( \sum_j P_{ij} V_{n-1}[1, j] \right) \max_{\alpha \in [0, 1]} \int_{-1}^{\infty} (1 + \alpha r)^\beta dG_i(r). \end{aligned}$$

By taking  $x = 1$  one sees that

$$V_n[1, i] = \left( \sum_j P_{ij} V_{n-1}[1, j] \right) \max_{\alpha \in [0, 1]} \int_{-1}^{\infty} (1 + \alpha r)^\beta dG_i(r)$$

and clearly  $V_n[x, i] = V_n[1, i] x^\beta$ .

Also the optimal fraction to be bet is determined by

$$\max_{\alpha \in [0, 1]} \int_{-1}^{\infty} (1 + \alpha r)^\beta dG_i(r)$$

which does not depend on  $x$  or  $n$  implying  $\alpha(i)$  does not depend on these parameters. ■

In the case  $\beta < 0$ , by considering the negativity of  $V_{n-1}[1,1]$  in the same way as in Theorem 2.3 below we conclude that the optimal  $\alpha$  is determined by

$$\min_{\alpha \in [0,1]} \int_{-1}^{\infty} (1 + \alpha r)^{\beta} dG_1(r) .$$

As a consequence of Theorem 2.2 we note that if  $V[x] = x$  (i.e.  $\beta = 1$ ) then one should always bet his current fortune, for then  $\alpha(i)$  is determined by

$$\max_{\alpha} \int_{-1}^{\infty} (1 + \alpha r) dG_1(r) = \max_{\alpha} (1 + \alpha ER_1)$$

if  $ER_1 > 0$  then the maximum is attained at  $\alpha = 1$ . It is interesting to note that it is precisely this strategy that maximizes the probability of ruin over any horizon!

Another utility function sometimes found in the literature is  $V[x] = 1 - e^{-x}$ , or, since adding a constant to the utility function does not alter the optimal strategy, its equivalent  $-e^{-x}$ . This utility does not yield simple strategies, unless the gambler is assumed to have infinite credit, that is, he is allowed to bet beyond his current fortune.

Theorem 2.3:

Assume that  $V[x] = -e^{-x}$  and that the gambler is allowed to wager any positive amount. Then  $V_n[x,i] = V_n[0,i]e^{-x}$  and the optimal wager  $y_n(x,i)$  is independent of  $n$  and  $x$ .

Proof:

$V_0[x,i] = -e^{-x}$  is clearly of the required form. Assume  $V_{n-1}[x,i] = V_{n-1}[0,i]e^{-x}$ . Since the utility is negative, we must have  $V_{n-1}[0,i] \leq 0$ . Now

$$V_n[x,i] = \max_{y \geq 0} \sum_j P_{ij} \int_{-1}^{\infty} V_{n-1}[0,j] e^{-(x+ry)} dG_i(r)$$

$$V_n[x,i] = e^{-x} \max_{y \geq 0} \left\{ \sum_j P_{ij} V_{n-1}[0,j] \int_{-1}^{\infty} e^{-ry} dG_i(r) \right\}.$$

Since  $V_n[0,j] \leq 0$  it follows that

$$\sum_j P_{ij} V_{n-1}[0,j] \leq 0.$$

Therefore

$$V_n[x,i] = e^{-x} \left( - \sum_j P_{ij} V_{n-1}[0,j] \right) \max_{y \geq 0} \left\{ - \int_{-1}^{\infty} e^{-ry} dG_i(r) \right\}$$

$$= e^{-x} \left( \sum_j P_{ij} V_{n-1}[0,j] \right) \min_{y \geq 0} \int_{-1}^{\infty} e^{-ry} dG_i(r).$$



By taking  $x = 0$  we have

$$V_n[0, i] = \left( \sum P_{ij} V_{n-1}[0, j] \right) \min_{y \geq 0} \int_{-1}^{\infty} e^{-ry} dG_i(r)$$

and thus

$$V_n[x, i] = V_n[0, i] e^{-x}.$$

We also note that the optimizing  $y$ , say  $y(i)$  is chosen from

$$\min_{y \geq 0} \int_{-1}^{\infty} e^{-ry} dG_i(r)$$

and does not depend on  $x$  or  $n$ . From Lemma 1.5 one concludes that this maximizing value is strictly positive. ■

As the following corollary points out, if the gambler is sufficiently wealthy, then, under this utility, he should always bet a fixed amount which depends only on the currently available gamble, even if no credit is available.

Corollary 2.4:

Assume that  $V[x] = -e^{-x}$  and that no credit is allowed. Let  $y(i)$  be as in Theorem 2.3 and let  $y^* = \sup y(i)$  assumed finite.

Then  $x \geq (n-1)y^* + y_i$  implies  $y_n(x, i) = y(i)$ .

Proof:

The proof hinges on the fact that  $x \geq (n-1)y^* + y(i)$  implies that betting  $y(j)$  whenever in state  $j$  will be feasible whatever the outcome of the  $n$  games in the planning horizon. Since this strategy is optimal in the (less restricted) situation where credit is available, its optimality follows. ■

We have thus seen that the optimal strategy for all these utility functions can be considered to depend essentially on the distribution of the gamble currently available. An interesting question that arises is how do these strategies depend on this gamble, that is, if I have a better gamble available, should I bet more? This intuitively appealing question is answered affirmatively (under some conditions) in the following theorems.

Theorem 2.5:

Assume the random variables  $R_i$  are stochastically increasing in  $i$ . Assume furthermore that  $V[x] = \text{Lnx}$ . Then the optimal proportion of the current fortune to be wagered,  $\alpha(i)$  is increasing in  $i$ .

Proof:

From the proof of Theorem 2.1 it follows that  $\alpha(i)$  is determined by

$$\max_{\alpha \in [0,1]} E \text{Ln}(1 + \alpha R_i) .$$

Since the set  $\{(\alpha, i) \mid \alpha \in [0, 1], i = 1, 2, 3, \dots\}$  is clearly a sublattice, the result will follow from the isotonicity theorem (A.11) if  $E\text{Ln}(1 + \alpha R_i)$  is superadditive on this sublattice, and, from Theorem A.3, this will be true if, for  $\epsilon > 0$  such that  $(\alpha + \epsilon) \in [0, 1]$ ,

$$E\text{Ln}(1 + (\alpha + \epsilon)R_i) - E\text{Ln}(1 + \alpha R_i) = E\text{Ln}\left(\frac{1 + (\alpha + \epsilon)R_i}{1 + \alpha R_i}\right)$$

is increasing in  $i$  for  $\alpha$  and  $\epsilon$  fixed.

Consider the function  $(1 + (\alpha + \epsilon)r)/(1 + \alpha r)$  which is clearly continuous and differentiable in  $r$ . Its derivative with respect to  $r$  is given by  $\frac{\epsilon}{(1 + \alpha r)^2} > 0$  so the function is increasing in  $r$ . Thus  $\text{Ln}\left(\frac{1 + (\alpha + \epsilon)r}{1 + \alpha r}\right)$  is also increasing in  $r$  and since the  $R_i$  are stochastically increasing in  $i$   $E\text{Ln}\left(\frac{1 + (\alpha + \epsilon)R_i}{1 + \alpha R_i}\right)$  is increasing in  $i$ . ■

The above result can be considered a generalization of Bellman and Kalaba's result for coin tossing games: in this case one should bet the fraction  $2p - 1$  where  $p$  is the probability of winning. Since Bernoulli random variables are stochastically increasing in  $p$  it is not surprising that their optimal fraction is also increasing in  $p$ .

#### Theorem 2.6:

Assume the random variables  $R_i$  are stochastically increasing in  $i$ . Furthermore assume that  $V[x] = x^\beta / \beta$   $\beta < 1$   $\beta \neq 0$ . Then,



calling  $\alpha(i)$  the optimal fraction of the current fortune to be wagered

- a) if  $\beta > 0$  then  $\alpha(i)$  is increasing in  $i$
- b) if  $\beta < 0$  and  $P\left[R_i > -\frac{1}{\beta}\right] = 0$  for all  $i$ , then  $\alpha(i)$  is increasing in  $i$ .

Proof:

In the case  $\beta > 0$ ,  $\alpha(i)$  is determined by

$$\max_{\alpha \in [0,1]} E(1 + \alpha R_i)^\beta$$

and, as in Theorem 2.5, will be increasing if  $E(1 + (\alpha + \epsilon)R_i)^\beta - E(1 + \alpha R_i)^\beta$  is increasing in  $i$  which will in turn follow if  $(1 + (\alpha + \epsilon)r)^\beta - (1 + \alpha r)^\beta$  is increasing in  $r$ . Clearly this is equivalent to showing that

$$\frac{\partial}{\partial \alpha} (1 + \alpha r)^\beta = \beta(1 + \alpha r)^{\beta-1} r$$

is increasing in  $r$ . To show this, we compute  $\frac{\partial}{\partial r} \beta(1 + \alpha r)^{\beta-1} r$  and show it is nonnegative.

$$\frac{\partial}{\partial r} \beta(1 + \alpha r)^{\beta-1} r = \beta(\beta - 1)(1 + \alpha r)^{\beta-2} \alpha r + \beta(1 + \alpha r)^{\beta-1} \geq 0$$

if and only if

$$(\beta - 1)\alpha r + 1 + \alpha r \geq 0$$

or equivalently

$$\beta \alpha r + 1 \geq 0$$

which follows from the fact that  $0 < \beta \leq 1$ ,  $0 \leq \alpha \leq 1$ ,  $R_i \geq -1$ .

In the case  $\beta < 0$ ,  $\alpha(i)$  is determined by

$$\min_{\alpha \in [0,1]} E(1 + \alpha R_i)^\beta.$$

By the remarks at the end of the appendix  $\alpha(i)$  will be increasing in  $i$  if  $E(1 + \alpha R_i)^\beta$  is subadditive over the sublattice of pairs  $(\alpha, i)$ , or, in other words, if, for  $\epsilon > 0$

$$E(1 + (\alpha + \epsilon) R_i)^\beta - E(1 + \alpha R_i)^\beta$$

is decreasing in  $i$ . This in turn will be true if the function

$$(1 + (\alpha + \epsilon)r)^\beta - (1 + \alpha r)^\beta$$

is decreasing in  $r$ , because of the stochastic monotonicity of  $R_i$ . Equivalently it suffices to show that

$$\frac{\partial}{\partial \alpha} (1 + \alpha r)^\beta = \beta r (1 + \alpha r)^{\beta-1}$$

is decreasing in  $r$ , that is, if its derivative with respect to  $r$  is negative

$$\beta(1 + \alpha r)^{\beta-1} + \beta(\beta - 1)\alpha r(1 + \alpha r)^{\beta-2} \leq 0$$

or, equivalently, since  $\beta < 0$

$$1 + \alpha r + (\beta - 1)\alpha r \geq 0$$

or

$$r \leq \left(-\frac{1}{\beta}\right)\frac{1}{\alpha}$$

which is true since by assumption

$$R_i \leq -\frac{1}{\beta} \leq \left(-\frac{1}{\beta}\right)\frac{1}{\alpha}$$

because  $\alpha \leq 1$ . ■

The intuitive reason behind the need for this upper bound on the outcome of the gamble in the case  $\beta < 0$  can best be explained by considering an example in which we assume the bound is violated.

Take  $\beta = -1$  and consider only gambles of the form:

$$R = \begin{cases} r & \text{with probability } p \\ -1 & \text{with probability } q = 1 - p \end{cases}$$

Assuming  $rp - q > 0$  (i.e.  $ER > 0$ ), the optimal  $\alpha$  is determined by

$$\min_{\alpha \in [0,1]} \left\{ \frac{p}{1 + \alpha r} + \frac{q}{1 - \alpha} \right\}$$

and is easily found to be given by

$$\alpha = \min \left\{ \frac{r - (1 + r)\sqrt{pqr}}{r(p - qr)}, 1 \right\}.$$

Now take  $p = .9$   $q = .1$ .



Gamble 1 has  $r = 2$  and  $\alpha = .5194$ .

Gamble 2 has  $r = 3$  and  $\alpha = .5119$ .

Clearly  $R_2$  is stochastically larger than  $R_1$  and yet it is optimal to bet less. The reason now is apparent: since the utility is bounded above (by 0), then betting more in gamble 2 than in gamble 1, if we win, does not increase our utility by much more, and if we loose, the decrease in utility is larger. Therefore we sacrifice some of the possible winnings in order to hedge against possible losses, showing once more a conservative behavior.

It is interesting to note that, in spite of the above, for coin tossing games, Bellman and Kalaba [1] find that

$$\alpha(i) = \frac{\frac{1}{p_i^{1-\beta}} - \frac{1}{q_i^{1-\beta}}}{\frac{1}{p_i^{1-\beta}} + \frac{1}{q_i^{1-\beta}}}$$

which is seen to be increasing in  $i$  if  $p_i$  is increasing in  $i$ .

For the utility function  $V[x] = -e^{-x}$ , it is easy to see that the amount  $y(i)$  to be wagered is not increasing in  $i$ , even if the  $R_i$  are stochastically increasing. In fact  $y(i)$  is determined by

$$\min_{y \geq 0} E e^{-R_i y}.$$

Assuming  $R_i$  has the same distribution as above,  $y(i)$  can be computed by

$$\min_{y \geq 0} \{pe^{-ry} + qe^y\}$$

and is easily seen to be given by

$$y = \max \{(\text{Lnr} + \text{Lnp} - \text{Lnq})/(1 + r), 0\}.$$

For the same gambles as before we obtain

$$y(1) = .9634$$

$$y(2) = .8240.$$

Hence it is evident that some conditions are needed to insure monotonicity.

Theorem 2.7:

Assume the random variables  $R_i$  are stochastically increasing in  $i$ . Assume furthermore that  $V[x] = -e^{-x}$ .

Suppose that the gambler has unlimited credit, or that his fortune is large, as per Corollary 2.4. Let  $y(i)$  be the optimal amount to wager and  $y^* = \sup y(i)$  assumed finite. Then if  $P\left[R_i > \frac{1}{y^*}\right] = 0 \forall i$   $y(i)$  is increasing in  $i$ .

Proof:

$y(i)$  is determined by  $\min_{y \geq 0} Ee^{-R_i y}$ . Since  $y(i) \leq y^*$  we can clearly reduce this problem to

$$\min_{0 \leq y \leq y^*} Ee^{-R_i y}.$$

$y(i)$  will be increasing in  $i$  if  $Ee^{-R_i y}$  is subadditive in  $y$  and  $i$ , that is if for  $\epsilon > 0$

$$Ee^{-R_i(y+\epsilon)} - Ee^{-R_i y}$$

is decreasing in  $i$ . This will follow by the stochastic monotonicity of  $R_i$  if  $e^{-r(y+\epsilon)} - e^{-ry}$  is decreasing in  $r$ , or equivalently if

$$\frac{\partial}{\partial y} e^{-ry} = -r e^{-ry}$$

is decreasing in  $r$ , which will follow if

$$\frac{\partial}{\partial r} (-r e^{-ry}) = -e^{-ry} + r y e^{-ry} \leq 0.$$

This will be true if  $r \leq \frac{1}{y}$  over the sublattice under consideration. But on this sublattice  $y \leq y^*$  implying  $\frac{1}{y^*} \leq \frac{1}{y}$  which in turn implies  $r \leq \frac{1}{y^*} \leq \frac{1}{y}$ . ■

Note that this bound depends on the distributions of  $R_i$ , and its computation requires the solution of the problem, which will answer directly the questions of monotonicity of  $y(i)$ . Thus the value of this result is only of theoretical interest.

For coin tossing games, one easily finds  $y(i) = \ln p_i - \ln q_i$  which is clearly increasing in  $i$  if  $p_i$  is.



## CHAPTER III

## DEPENDENCE OF THE OPTIMAL STRATEGY ON THE FORTUNE

We now return to the general problem of Chapter I and discuss the dependence of the optimal amount to bet,  $y_n(x, i)$  on the gambler's fortune  $x$ .

From the results of the previous chapter, it is tempting to postulate the generalizations

- a)  $y_n(x, i)$  is increasing in  $x$ , or
- b)  $\frac{1}{x} y_n(x, i)$  is decreasing in  $x$ .

Both of these are false, as shown by the following example.

Let

$$V[x] = \begin{cases} x - x^2/2 & 0 \leq x \leq .8 \\ .32 + .2x & .8 < x \end{cases}$$

Let the gamble be a coin tossing game with probability .6 of winning.

It is an easy matter to compute the optimal strategy for one stage horizon as

$$y_1(x) = \begin{cases} x & 0 \leq x \leq 1/6 \\ .2(1 - x) & 1/6 < x \leq 3/4 \\ x - .7 & 3/4 < x \end{cases}$$

which is clearly not monotone in  $x$ .

The optimal fraction to bet is

$$\alpha(x) = \frac{1}{x} y_1(x) = \begin{cases} 1 & 0 < x \leq 1/6 \\ .2 \frac{1}{x} - 1 & 1/6 < x \leq 3/4 \\ 1 - \frac{.7}{x} & 3/4 < x \end{cases}$$

which is clearly not monotone in  $x$ .

However, under some conditions, we can find functions that depend on the optimal policy and exhibit simple monotone behaviour.

Let  $R_i$  be a random variable  $P[R_i \geq -1] = 1$ . Define  $\rho_i = \sup \{r : P[0 < R_i \leq r] = 0\}$ . In other words  $R_i$  has probability 0 of being strictly positive and smaller than  $\rho_i$  and  $\rho_i$  is the largest of all such values.

Theorem 3.1:

Assume the state of gambling system is  $i$ . Assume  $\rho_i > 0$ . Then, if  $y_n(x, i)$  is the optimal amount to bet  $x + \rho_i y_n(x, i)$  is increasing in  $x$ .

Proof:

$$V_n[x, i] = \max_{0 \leq y \leq x} E_{R, J/i} V_{n-1}[x + Ry, J].$$

For  $y \geq 0$ , let  $v = x + \rho_i y$ . Then  $y \in [0, x]$  implies  $v \in [x, x(1 + \rho_i)]$  and we have

$$V_n[x, i] = \max_v E_{R, J/i} V_{n-1} \left[ x \left( 1 - \frac{R}{\rho_i} \right) + \frac{R}{\rho_i} v, J \right]$$

subject to  $x \leq v \leq x(1 + \rho_i)$ .

Note this constraint set is a sublattice of pairs  $(x, v)$  by

Examples A1.4 and Lemma A.1.

Now consider  $V_{n-1}\left[x\left(1 - \frac{R}{\rho_1}\right) + \frac{R}{\rho_1} v, j\right]$ . For  $R \leq 0$   
 $\left(1 - \frac{R}{\rho_1}\right) \geq 0$  and  $\frac{R}{\rho_1} \leq 0$ . Thus in this case  $V_{n-1}\left[x\left(1 - \frac{R}{\rho_1}\right) + \frac{R}{\rho_1} v, j\right]$  is superadditive in  $x$  and  $v$  on our sublattice, because  $V_{n-1}[x, j]$  is concave in  $x$  and Example A2.5 applies.

For  $R > 0$ , we need only consider  $R \geq \rho_1$  by the definition of  $\rho_1$ . In this case  $\frac{R}{\rho_1} \geq 1$  and thus

$$V_{n-1}\left[x\left(1 - \frac{R}{\rho_1}\right) + \frac{R}{\rho_1} v, j\right] = V_{n-1}\left[\frac{R}{\rho_1} v - \left|1 - \frac{R}{\rho_1}\right| x, j\right]$$

again superadditive in  $v$  and  $x$  by the same reasons. Since mixtures of superadditive functions are clearly superadditive, we have that

$$E_{R,j/i} V_{n-1}\left[x\left(1 - \frac{R}{\rho_1}\right) + \frac{R}{\rho_1} v, j\right]$$

is superadditive in  $x$  and  $v$  on the sublattice

$$\{(x, v) \mid x \leq v \leq x(1 + \rho_1)\}.$$

Thus, by the isotonicity theorem (A.11)  $v_n(x, i)$ , the optimizing value is increasing in  $x$ . This optimal value is clearly

$$v_n(x, i) = x + \rho_1 y_n(x, i). \blacksquare$$



Note that if the optimizing value of  $y$  is not unique (in which case there must be an interval of maximizing values because of the concavity of the function being maximized) then the isotonicity theorem guarantees that we may choose the largest of such values, or the smallest and, as long as we are consistent in this choice, the result will still hold.

The function given is tight, in the sense that, if  $\rho_i > 0$  and  $\rho > \rho_i$  then there exist problems for which  $x + \rho y_n(x, i)$  is not increasing in  $x$ , as the following examples shows.

Let

$$V[x] = \begin{cases} 2x & x \leq 4 \\ 6 + \frac{1}{2}x & 4 < x \end{cases}$$

Assume only one stage and let the gamble available be given by

$$R = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -\frac{1}{2} & \text{with probability } \frac{1}{2} \end{cases}$$

It is easy to see that the optimal betting strategy is

$$y_1(x, i) = \begin{cases} x & \text{if } 0 \leq x \leq 2 \\ 4 - x & \text{if } 2 < x \leq 4 \\ 2x - 8 & \text{if } 4 < x \leq 8 \\ x & \text{if } 8 < x \end{cases}$$

Clearly  $\rho_i = 1$ . For  $\rho > 1$ , in the region  $2 < x \leq 4$

$x + \rho y_1(x, i) = x(1 - \rho) + 4\rho$  decreasing in  $x$ .

Hence our result is tight in the sense described above.

The decision variable we have considered so far is the amount to bet  $y_n(x, i)$ . An alternative would be to decide how much not to bet, and wager the remainder.

Let  $u_n(x, i) = x - y_n(x, i)$ . Examining the results of the previous chapter, one is tempted to conclude that this quantity is an increasing function of  $x$ . This is not generally true, as can be seen from the last example. In the region  $4 < x \leq 8$   $u_1(x, i) = 8 - x$ , decreasing in  $x$ .

However, in the same spirit of Theorem 3.1 we have the following result.

Let  $R_i$  be a random variable  $P[R_i \geq -1] = 1$ . Define  $\xi_i = |\inf \{r : P[r \leq R_i < 0] = 0\}|$ .

In other words  $R_i$  has probability 0 of being strictly negative and larger than  $-\xi_i$  and  $\xi_i$  is the largest of all such values.  $0 \leq \xi_i \leq 1$  by definition, if it exists.

### Theorem 3.2:

Assume the state of the gambling system is  $i$ .

Assume  $\xi_i$  exists and  $\xi_i > 0$ . Then if  $y_n(x, i)$  is the optimal amount to bet,  $x - \xi_i y_n(x, i)$  is increasing in  $x$ .

### Proof:

From the optimality equation

$$V_n[x, i] = \max_{0 \leq y \leq x} E_{R, J/i} V_{n-1}[x + Ry, J].$$

By making the change of variables  $u = x - \xi_i y$  we have

$$V_n[x, i] = \max_u E_{RJ/i} V_{n-1} \left[ x \left( 1 + \frac{R}{\xi_i} \right) - \frac{R}{\xi_i} u, j \right]$$

$$\text{subject to } x(1 - \xi_i) \leq u \leq x.$$

Note this set is a sublattice in the pair  $(x, u)$  by Example A1.6

and Lemma A.1.

Now consider  $V_{n-1} \left[ x \left( 1 + \frac{R}{\xi_i} \right) - \frac{R}{\xi_i} u, j \right]$ . For  $R \geq 0$   $1 + \frac{R}{\xi_i} \geq 0$   $\frac{R}{\xi_i} \geq 0$  and thus, since  $V_{n-1}[x, j]$  is concave in  $x$ ,  $V_{n-1} \left[ x \left( 1 + \frac{R}{\xi_i} \right) - \frac{R}{\xi_i} u, j \right]$  is superadditive in  $x$  and  $u$  by Example A2.5.

For  $R < 0$ , we need only consider those  $R$  satisfying

$$-1 \leq R \leq -\xi_i, \text{ because of the definition of } \xi_i.$$

Then  $\frac{R}{\xi_i} \leq -1$  and thus

$$V_{n-1} \left[ x \left( 1 + \frac{R}{\xi_i} \right) - \frac{R}{\xi_i} u, j \right] = V_{n-1} \left[ \left| \frac{R}{\xi_i} \right| u - \left| 1 + \frac{R}{\xi_i} \right| x, j \right]$$

is again superadditive in  $x$  and  $u$ .

Since mixtures of superadditive functions are superadditive we have  $E_{RJ/i} V_{n-1} \left[ x \left( 1 + \frac{R}{\xi_i} \right) - \frac{R}{\xi_i} u, j \right]$  is superadditive in  $x$  and  $u$ , on the sublattice  $x(1 - \xi_i) \leq u \leq x$ . Thus, by the isotonicity theorem (A.11)  $u_n(x, i)$ , the optimizing value, is increasing in  $x$ . This optimal value is clearly  $u_n(x, i) = x - \xi_i y_n(x, i)$ . ■

The remark that follows Theorem 3.1 clearly applies here as well.

The monotonicity property of  $x - \xi_i y_n(x, i)$  is tight, in the same sense as that of  $x + \rho_i y_n(x, i)$  as can be seen from the example preceeding this theorem.



There  $\xi_1 = \frac{1}{2}$ . For  $\xi > \frac{1}{2}$ , in the region  $4 < x \leq 8$   
 $x - \xi y_1(x, i) = x(1 - 2\xi) + 8\xi$  decreasing in  $x$ .

It is interesting to note that the proofs of Theorems 3.1 and 3.2 do not depend on the allowable values of  $x$  and  $y$ . Thus, even if we are constrained to integer fortunes and integer wagers, the results will still be valid.

The combination of Theorems 3.1 and 3.2 also yields the following result, useful in the actual computation of optimal policies.

Corollary 3.3:

Assume  $\rho_1 > 0$  and  $\xi_1$  exists  $\xi_1 > 0$ . Then, for  $\epsilon > 0$

$$\frac{\epsilon}{\xi_1} \geq y_n(x + \epsilon, i) - y_n(x, i) \geq -\frac{\epsilon}{\rho_1}.$$

Proof:

$x + \rho_1 y_n(x, i)$  is increasing in  $x$  so

$$x + \epsilon + \rho_1 y_n(x + \epsilon, i) \geq x + \rho_1 y_n(x, i)$$

implying

$$y_n(x + \epsilon, i) - y_n(x, i) \geq -\frac{\epsilon}{\rho_1}.$$

Similarly  $x - \xi_1 y_n(x, i)$  is increasing in  $x$  so

$$x + \epsilon - \xi_1 y_n(x + \epsilon, i) \geq x - \xi_1 y_n(x, i)$$

implying

$$\frac{\epsilon}{\xi_1} \geq y_n(x + \epsilon, i) - y_n(x, i) \quad \blacksquare$$

From this result, it is immediate that

Corollary 3.4:

Assume  $\rho_i > 0$  and  $\xi_i$  exists and  $\xi_i > 0$ . Then  $y_n(x, i)$  is uniformly continuous in  $x$ .

The specialization of Corollary 3.3 to coin tossing games will prove useful in the sequel, and is interesting in its own right.

In this case  $\rho_i = \xi_i = 1$  hence  $x + y_n(x, i)$  is increasing in  $x$ : "The richer you are, the more you strive for."

$x - y_n(x, i)$  is increasing in  $x$ : "The richer you are, the more you save."

Corollary 3.5:

In coin tossing games  $|y_n(x + \epsilon, i) - y_n(x, i)| \leq \epsilon$ .

## CHAPTER IV

## DEPENDENCE OF THE OPTIMAL STRATEGY ON THE GAMBLE

We now consider the effect of the gamble on the betting strategy.

From the examples of Chapter II, it is clear that assuming the random variables are stochastically increasing in  $i$ , does not guarantee that  $y_n(x,i)$  is increasing in  $i$ . In fact, the examples show that, even if one gambler is allowed to play the same game over all his horizon, and another gambler is allowed to play a stochastically better game for the same number of periods, it does not follow that the second gambler should wager more, if their fortunes are equal.

It is thus evident that we must restrict the class of gambles considered, in order to obtain meaningful results. In the class of games considered by Ferguson [7], the gambler faces at each stage a coin tossing game, with probability of winning  $p_i$ , known to him. However at each stage the  $p_i$  is drawn from some distribution  $F(p)$ , independently of the outcome of the bet and of the previous values of  $p$ . The question of whether one should bet more with a higher  $p$  is answered affirmatively in the following.

Proposition 4.1:

Let  $y_n(x,p)$  be the optimal wager when you have  $n$  plays available, your fortune is  $x$  and you face a coin toss with probability  $p$  of winning. Assume at each stage  $p$  is independently drawn from a distribution  $F(p)$ . Then  $y_n(x,p)$  is increasing in  $p$ .



Proof:

By identifying the state of the gambling system,  $i$ , with the probability of winning, we have

$$V_n[x, p] = \max_{0 \leq y \leq x} \{pEV_{n-1}[x + y, T] + (1 - p)EV_{n-1}[x - y, T]\}$$

where  $T$  is a random variable with distribution  $F$ . Since, for fixed  $x$ , the set  $\{(y, p) : 0 \leq y \leq x, 0 \leq p \leq 1\}$  is a sublattice, we only need to show the superadditivity of the function being maximized.

Let

$$\phi(y, p) = pEV_{n-1}[x + y, T] + (1 - p)EV_{n-1}[x - y, T],$$

by Theorem A3, we have to show that for  $\epsilon > 0$

$$\phi(y, p + \epsilon) - \phi(y, p)$$

is increasing in  $y$ , or equivalently  $\frac{\partial}{\partial p} \phi(y, p)$  is increasing in  $y$

$$\frac{\partial}{\partial p} \phi(y, p) = EV_{n-1}[x + y, T] - EV_{n-1}[x - y, T]$$

which follows from the fact that  $V_{n-1}[x, p]$  is increasing in  $x$ . ■

Another interesting question that arises is that of two gamblers with the same utility function, both facing a coin tossing game in which their probabilities of winning are constant over their horizon but different from each other. It is intuitively appealing that the one with the higher probability of winning should bet more. The

direct proof of this is, however, surprisingly difficult, and our indirect proof yields also an answer to another question of interest.

Suppose that  $n$  different coin tossing games are available, with probabilities of winning  $p_1, p_2, \dots, p_n$ . You are allowed to play each game once, but can choose the order in which you will play them. The question of which is the optimal order has its rather surprising answer in the following.

Theorem 4.2:

In the game described above, the value of the optimal expected utility  $V_n[x]$  is independent of the order in which the  $p_i$  are chosen.

Proof:

Assume  $n = 2$ . The optimal strategy will consist of the following information

- a) The game to be played first
- b) The bet to be placed in the first game, say  $y$
- c) The bet to be placed in the second game if we win the first one, say  $y_W$
- d) The bet to be placed in the second game if we loose the first one, say  $y_L$ .

We will prove the result by showing that choosing  $p_1$  first and then continuing optimally yields the same expected utility that can be obtained by choosing  $p_2$  first and then continuing optimally.

Let  $q_1 = 1 - p_1$ .

Assume we play  $p_1$  first and  $p_2$  second. Then, conditioning on the outcome of the tosses, our expected utility is

$$p_1 p_2 V[x + y + y_W] + p_1 q_2 V[x + y - y_W] + \\ q_1 p_2 V[x - y + y_L] + q_1 q_2 V[x - y - y_L]$$

and clearly the optimal betting strategy is determined by solving the nonlinear problem

$$\begin{aligned} \max_{y, y_W, y_L} \{ & p_1 p_2 V[x + y + y_W] + p_1 q_2 V[x + y - y_W] + \\ & q_1 p_2 V[x - y + y_L] + q_1 q_2 V[x - y - y_L] \} \\ \text{subject to } & x \geq y \geq 0 \\ & x + y \geq y_W \geq 0 \\ & x - y \geq y_L \geq 0. \end{aligned}$$

Let  $y^*$ ,  $y_W^*$  and  $y_L^*$  be the optimal strategy.

Similarly, if we decide to play  $p_2$  first and then  $p_1$ , we have to solve

$$\begin{aligned} \max_{y, y_W, y_L} \{ & p_2 p_1 V[x + y + y_W] + p_2 q_1 V[x + y - y_W] + \\ & q_2 p_1 V[x - y + y_L] + q_2 q_1 V[x - y - y_L] \} \\ \text{subject to } & x \geq y \geq 0 \\ & x + y \geq y_W \geq 0 \\ & x - y \geq y_L \geq 0. \end{aligned}$$



Define

$$\begin{aligned}
 y^+ &= \frac{1}{2} (y_W^* + y_L^*) \\
 (4.1) \quad y_W^+ &= y^* + \frac{1}{2} (y_W^* - y_L^*) \\
 y_L^+ &= y^* - \frac{1}{2} (y_W^* - y_L^*).
 \end{aligned}$$

We will show that these values are feasible for the second problem, and yield the same value of the objective function as was optimal for the first problem. Hence the optimal value of the second problem is greater than that of the first, and playing  $p_2$  first is at least as good as playing  $p_1$  first. Reversing the argument (since we have not imposed conditions on the  $p_i$ ) shows both maxima are the same and hence the order in which the  $p_i$  are chosen is irrelevant.

Feasibility of  $y^+$ ,  $y_W^+$  and  $y_L^+$ :

$$y_W^* \geq 0, y_L^* \geq 0 \Rightarrow y^+ \geq 0$$

$$x + y^* \geq y_W^*, x - y^* \geq y_L^* \Rightarrow x \geq \frac{1}{2} (y_W^* + y_L^*) = y^+$$

$$\begin{aligned}
 x + y^+ - y_W^+ &= x + \frac{1}{2} (y_W^* + y_L^*) - y^* - \frac{1}{2} (y_W^* - y_L^*) \\
 &= x - y^* + y_L^* \geq 0
 \end{aligned}$$

$$\begin{aligned}
 x - y^+ - y_L^+ &= x - \frac{1}{2} (y_W^* + y_L^*) - y^* + \frac{1}{2} (y_W^* - y_L^*) \\
 &= x - y^* - y_L^* \geq 0.
 \end{aligned}$$

Also  $y_W^*$  is the amount to bet when there is one stage left and our fortune is  $x + y^*$  (since we have won our bet of  $y^*$  is the previous stage). Hence

$$y_W^* = y_1(x + y^*, p_2).$$

$$\text{Similarly } y_L^* = y_1(x - y^*, p_2).$$

Thus by Corollary 3.5

$$|y_W^* - y_L^*| \leq 2y^*$$

showing  $y^* \pm \frac{1}{2}(y_W^* - y_L^*) \geq 0$  i.e.  $y_W^+ \geq 0$ ,  $y_L^+ \geq 0$ .

Hence  $y^+$ ,  $y_W^+$ ,  $y_L^+$  satisfy the constraints of the second problem. By substituting them in its objective function one obtains

$$\begin{aligned} & p_2 p_1 V[x + y^+ + y_W^+] + p_2 q_1 V[x + y^+ - y_W^+] + \\ & q_2 p_1 V[x - y^+ + y_L^+] + q_2 q_1 V[x - y^+ - y_L^+] = \\ & p_1 p_2 V[x + y^* + y_W^*] + p_2 q_1 V[x - y^* + y_L^*] + \\ & q_2 p_1 V[x + y^* - y_W^*] + q_2 q_1 V[x - y^* - y_L^*] \end{aligned}$$

which was the optimal value in the first problem. It now remains only to complete the proof for  $n > 2$ .

Suppose in an  $(n - 1)$  stage problem the order of the  $p_i$  is irrelevant. Consider an  $n$  stage problem. Choose  $p_1$  to be the first and then continue optimally with  $p_2, p_3, \dots, p_n$ . This is equivalent to choosing  $p_1$  and playing a one stage game where the final utility is that given by the optimal value function of the  $n - 1$  stage problem with probabilities  $p_2, p_3, \dots, p_n$ .

Thus we can choose any one of these, say  $p_2$ , to be second, and continue with  $p_3, p_4, \dots, p_n$ .

But the first two stages can be thought of as a two stage problem whose final utility is the optimal value function of the  $n - 2$  stage game with  $p_3, p_4, \dots, p_n$ . Thus in this two stage game we can change the order and play  $p_2$  first then  $p_1$  without affecting the overall optimal value. Thus choosing  $p_2$  first is at least as good as choosing  $p_1$  first and since the numbering of the  $p_i$  was arbitrary the result follows. ■

Now consider a gambler who faces  $n$  tosses of the same biased coin, with probability  $p > \frac{1}{2}$  of winning.

Let  $y_n(x, p)$  be his optimal wager. We will study the dependence of  $y_n(x, p)$  on  $p$ .

First we have an interesting functional equation satisfied by  $y_n(x, p)$ , which says that the optimal wager is the average of the optimal wagers to be placed upon loosing and upon winning.

Proposition 4.3:

$$y_n(x, p) = \frac{1}{2} \{y_{n-1}(x + y_n(x, p), p) + y_{n-1}(x - y_n(x, p), p)\}.$$

Proof:

In the game of Theorem 4.2 set  $p_1 = p_2 = p_3 = \dots p_n = p$ . Then exchanging  $p_1$  and  $p_2$  does not change the optimal policy. Hence  $y^+ = y^*$  and Equations (4.1) apply. Noting that  $y^* = y_n(x, p)$ ,  $y_W^* = y_{n-1}(x + y_n(x, p), p)$ ,  $y_L^* = y_{n-1}(x - y_n(x, p), p)$ , the result follows from the first of these equations. ■

An interesting property of this coin tossing game is the following.



Lemma 4.4:

Assume that for some  $x > 0$   $y_n(x, p) = x$ . If  $V[\cdot]$  is strictly increasing and differentiable, then, for all  $x' \in [0, x]$   $y_n[x', p] = x'$ .

Proof:

By Theorem 1.6  $V_n[\cdot, p]$  is strictly increasing and differentiable. By Theorem 1.2 it is concave.

$y_n(x, p)$  is determined by

$$\max_{0 \leq y \leq x} \{pV_{n-1}[x+y, p] + (1-p)V_{n-1}[x-y, p]\}.$$

By the concavity of  $V_{n-1}[\cdot, p]$ ,  $y_n(x, p) = x$  if and only if

$$pV'_{n-1}[2x, p] - (1-p)V'_{n-1}[0, p] \geq 0.$$

Now, for  $x' \in [0, x]$

$$pV'_{n-1}[2x', p] - (1-p)V'_{n-1}[0, p] \geq pV'_{n-1}[2x, p] - (1-p)V'_{n-1}[0, p] \geq 0$$

since  $V'_{n-1}[\cdot, \cdot]$  is decreasing in the first argument by concavity.

Hence  $y_n(x', p) = x'$ . ■

We can now prove

Theorem 4.5:

Assume  $V$  is strictly increasing and differentiable. Then  $y_n(x, p)$  is increasing in  $p$ , and  $y_n(x, p) < x$  implies  $y_n(x, p) < y_n(x, p + \delta)$  for  $0 < \delta \leq 1 - p$ .

Proof:

We first show that these properties hold for  $n = 1$ .  $y_1(x, p)$  is determined by

$$\max_{0 \leq y \leq x} \{pV(x+y) + qV(x-y)\}.$$

If  $y_1(x, p) = x$  then, since  $V$  is differentiable

$$pV'(2x) - qV'(0) \geq 0.$$

But then, for  $\epsilon > 0$

$$(p + \epsilon)V'(2x) - (q - \epsilon)V'(0) = pV'(2x) - qV'(0) + \epsilon[V'(2x) + V'(0)] > 0$$

hence with  $p + \epsilon$  we also get  $x$ .

If  $0 < y_1(x, p) < x$  then

$$pV'[x + y_1(x, p)] - qV'[x - y_1(x, p)] = 0$$

implying for  $\epsilon > 0$

$$\begin{aligned} (p + \epsilon)V'[x + y_1(x, p)] - (q - \epsilon)V'[x - y_1(x, p)] &= \\ pV'[x + y_1(x, p)] - qV'[x - y_1(x, p)] + \\ \epsilon \{V'[x + y_1(x, p)] + V'[x - y_1(x, p)]\} &= \\ \epsilon \{V'[x + y_1(x, p)] + V'[x - y_1(x, p)]\} &> 0 \end{aligned}$$

since  $V(\cdot)$  is strictly increasing. Hence

$$y_1(x, p + \epsilon) > y_1(x, p).$$

The case  $y_1(x, p) = 0$  does not arise since

$$pV'[x] - qV'[x] = (p - q)V'[x] > 0$$

because  $p > \frac{1}{2}$  and  $V(\cdot)$  is strictly increasing. For  $n > 1$ .

Choose  $p > \frac{1}{2}$  and pick  $p^+ = p + \epsilon > p$ .

Let  $\bar{y} = y_n(x, p)$  and  $\bar{y}^+ = y_n(x, p^+)$ . Assume, as an inductive hypothesis, that  $y_{n-1}(x, p)$  satisfies the properties we are trying to establish. Proposition 4.3 states that

$$\begin{aligned}\bar{y} &= \frac{1}{2} \left\{ y_{n-1}[x + \bar{y}, p] + y_{n-1}[x - \bar{y}, p] \right\} \\ \bar{y}^+ &= \frac{1}{2} \left\{ y_{n-1}[x + \bar{y}^+, p^+] + y_{n-1}[x - \bar{y}^+, p^+] \right\} . \\ \bar{y} - \bar{y}^+ &= \frac{1}{2} \left\{ y_{n-1}(x + \bar{y}, p) + y_{n-1}(x - \bar{y}, p) - y_{n-1}(x + \bar{y}^+, p^+) - \right. \\ &\quad \left. y_{n-1}(x - \bar{y}^+, p^+) \right\} .\end{aligned}$$

If  $y_{n-1}(x + \bar{y}, p) < x + \bar{y}$  (implying  $\bar{y} < x$ ) then since  $y_{n-1}(x + \bar{y}, p)$  is strictly increasing in  $p$

$$\begin{aligned}\bar{y} - \bar{y}^+ &< \frac{1}{2} \left\{ y_{n-1}(x + \bar{y}, p^+) + y_{n-1}(x - \bar{y}, p^+) - \right. \\ &\quad \left. y_{n-1}(x + \bar{y}^+, p^+) - y_{n-1}(x - \bar{y}^+, p^+) \right\} .\end{aligned}$$

Now Corollary 3.5 states

$$\begin{aligned}y_{n-1}(x + \bar{y}, p^+) - y_{n-1}(x + \bar{y}^+, p^+) &\leq |\bar{y} - \bar{y}^+| \\ y_{n-1}(x - \bar{y}, p^+) - y_{n-1}(x - \bar{y}^+, p^+) &\leq |\bar{y} - \bar{y}^+| .\end{aligned}$$

Hence  $\bar{y} - \bar{y}^+ < |\bar{y} - \bar{y}^+|$  implying  $\bar{y} - \bar{y}^+ < 0$ .



If  $y_{n-1}(x + \bar{y}^+, p^+) < x + \bar{y}^+$  (implying  $\bar{y}^+ < x$ ) then since  $y_{n-1}(x + \bar{y}^+, p^+)$  is strictly increasing in  $p^+$

$$\bar{y} - \bar{y}^+ < \frac{1}{2} \left\{ y_{n-1}(x + \bar{y}, p) + y_{n-1}(x - \bar{y}, p) - \right.$$

$$\left. y_{n-1}(x + \bar{y}^+, p) - y_{n-1}(x - \bar{y}^+, p) \right\}$$

and the same argument applies, showing  $\bar{y} < \bar{y}^+$ . In the case

$y_{n-1}(x + \bar{y}, p) = x + \bar{y}$  and  $y_{n-1}(x + \bar{y}^+, p^+) = x + \bar{y}^+$  then, by

Lemma 4.4  $y_{n-1}(x - \bar{y}, p) = x - \bar{y}$  and  $y_{n-1}(x - \bar{y}^+, p^+) = x - \bar{y}^+$ .

Hence  $\bar{y} = x$   $\bar{y}^+ = x$ , completing the proof. ■

## CHAPTER V

## GENERALIZATION TO ADAPTIVE MODELS

In many gambling situations the player does not know the exact distribution of the random variables that determine the outcome of the game. A prime example are the slot machines of Nevada casinos, where the probabilities of the different possible outcomes are a closely guarded secret. In such cases, the best the gambler can hope for is to obtain information on the distribution as the game progresses. This accumulation of information and subsequent modification of behaviour is usually termed adaptive behaviour.

We will assume initially a fairly general scenario, which will later be restricted to coin tossing games.

Assume a gambler with an initial fortune  $x$  faces  $n$  realizations of a game. If he bets an amount  $y \in [0, x]$  his fortune will change to  $x + Ry$  where  $R$  is a random variable. He knows also that the distribution of  $R$  is the same throughout all  $n$  plays. Furthermore he knows the distribution is a member of the set  $\{G_t : t \geq 0\}$ , although he does not know with certainty which one it is.

From past experience he has formulated a prior distribution  $F(\cdot)$  with the interpretation that, if  $G_T$  is the distribution he will be using then

$$F(t) = P[T \leq t] .$$

After observing the outcome of the first play, the gambler computes in the usual manner, a posterior distribution on  $T$ , given by the conditional distribution

$$\tilde{F}_r(t) = P[T \leq t/R = r]$$

and he then continues the game, with a new fortune and a new prior distribution.

We will assume that the gambler wishes to play in such a way as to maximize the expected value of the utility  $V$ , of his final fortune, where his utility function possesses the properties discussed in Chapter I.

We will also assume that there exists an upper bound on the possible outcomes of the gamble, that is, we will assume there exists a  $\theta$ ,  $1 < \theta < \infty$  such that  $G_t(\theta) = 1$  for all  $t$ .

If the gambler decides not to bet at any stage of the game, then we will assume that the game is "played" nevertheless, and the gambler observes the outcome of the game, thus gaining information without risk.

Let  $V_n[x, F]$  be the supremum expected utility for a gambler who starts out with a fortune of  $x$ , a prior distribution  $F$  and  $n$  stages to go.

Then the usual dynamic programming formalism shows that

$$V_n[x, F] = \sup_{0 \leq y \leq x} \left\{ \int_{-1}^{\infty} V_{n-1}[x + ry, \tilde{F}_r] dG(r) \right\}$$

$$V_0[x, F] = V[x]$$

where



$$G = \int_0^{\infty} G_t dF(t) ,$$

and  $\tilde{F}_R$  is as defined above.

We must now proceed to prove the existence of an optimal policy, and the properties of the optimal value function  $V_n[x, F]$ .

Lemma 5.1:

If  $V_{n-1}[x, F]$  is continuous in  $x$  on  $(0, \infty)$  and right continuous at the origin for all  $F$ , then there exists a number  $y_n(x, F)$  such that

$$V_n[x, F] = \int_{-1}^{\infty} V_{n-1}[x + ry_n(x, F), \tilde{F}_r] dG(r) .$$

Theorem 5.2:

For all  $F$ ,  $V_n[x, F]$  is increasing and concave in  $x$ . It is thus continuous in  $x$  on  $(0, \infty)$ . Furthermore  $V_n[x, F] \leq V[x(1 + \theta)^n]$ , and it is right continuous at the origin. Hence an optimal policy exists.

From this theorem it follows that the supremum in the optimality equation is always attained, and we can, and will write it as

$$V_n[x, F] = \max_{0 \leq y \leq x} \int_{-1}^{\infty} V_{n-1}[x + ry, \tilde{F}_r] dG(r)$$

$$V_0[x, F] = V[x] .$$

We will call  $y_n(x, F)$  the optimal amount to wager and if not unique we will assume that either the smallest or the largest is chosen, as long as the gambler is consistent in his choice.

The proof of the above is identical to that of Lemma 1.1 and Theorem 1.2, except that, in this case, due to the definition of  $\theta$ , it is immediate that

$$V_n[x, F] \leq V[x(1 + \theta)^n]$$

since, as you cannot win more than  $\theta$  in any given play, the best you can hope for is to win all games, and to have wagered all your fortune at each toss, and your fortune will then be smaller than  $x(1 + \theta)^n$ .

The proof of Proposition 1.3 shows

Proposition 5.3:

$V_n[x, F]$  is increasing in  $n$ .

A rather surprising fact is that, in the adaptive model, it is sometimes optimal to bet a positive amount in unfavourable games, as the following example shows.

Consider a coin tossing game. The probability of winning is known to be either 1 or 0 with a prior probability .4 of being 1. One desires to maximize his expected fortune (i.e.  $V[x] = x$ ) in two stages.

It is easy to see that the optimal strategy is: bet  $x$ . If you win, bet  $2x$ . If you lose bet nothing.

Thus, although in the first toss the probability of winning is .4, one should wager all his fortune.

The special utilities  $V[x] = \ln(x)$  and  $V[x] = \frac{x^\beta}{\beta}$   $\beta < 1$   $\beta \neq 0$  yield simple strategies.

Theorem 5.4:

If  $V[x] = \text{Lnx}$  then

$$V_n[x, F] = V_n[1, F] + \text{Lnx}$$

$$y_n[x, F] = x\alpha(F)$$

where  $\alpha(F)$  does not depend on  $n$  or  $x$ .

If  $V[x] = \frac{x^\beta}{\beta}$   $\beta \leq 1$ ,  $\beta \neq 0$  then

$$V_n[x, F] = V_n[1, F]x^\beta$$

$$y_n[x, F] = x\alpha_n(F)$$

where  $\alpha_n(F)$  does not depend on  $x$ .

The proof of this theorem is identical to that of Theorems 2.1 and 2.2, and is hence omitted.

The dependence of  $y_n(x, F)$  on  $x$  is identical to that in the nonadaptive case.

As defined in Chapter III, let

$$\rho_F = \sup \{r : r \geq 0, G(r) = G(0)\}$$

$$\xi_F = |\inf \{r : r \leq 0, G(0) = G(r)\}|$$

where  $G = \int G_t dF(t)$ .

Theorem 5.5:

Let  $y_n(x, F)$  be the optimal amount to wager. If  $\rho_F > 0$  then  $x + \rho_F y_n(x, F)$  is increasing in  $x$ . If  $\xi_F > 0$  then  $x - \xi_F y_n(x, F)$  is increasing in  $x$ .



The proof of this theorem is identical to that of Theorems 3.1 and 3.2 and is hence omitted.

From this theorem there immediately follows

Corollary 5.6:

If both  $\rho_F > 0$  and  $\xi_F > 0$  then, for  $\epsilon > 0$

$$\frac{\epsilon}{\xi_F} \geq y_n(x + \epsilon, F) - y_n(x, F) \geq -\frac{\epsilon}{\rho_1}$$

hence  $y_n(x, F)$  is uniformly continuous in  $x$ .

Corollary 5.7:

In coin tossing games  $|y_n(x + \epsilon, F) - y_n(x, F)| \leq \epsilon$ .

The proofs are of course repetition of those of Corollaries 3.3, 3.4, 3.5.

We now turn to an examination of the dependence of both  $V_n(x, F)$  and  $y_n(x, F)$  on the prior distribution  $F$ . As in the case of nonadaptive models, we will limit ourselves to coin tossing games.

In this case, each distribution  $G_i$ ,  $i = 1, 2, \dots$  is characterized by a probability  $p_i$  of winning. The prior distribution  $F$  is characterized by a sequence  $\{\pi_1, \pi_2, \dots\}$  with the interpretation that the game we are playing will have probability  $p_i$  of a win with probability  $\pi_i$ , that is, we will be playing against coin  $i$  with probability  $\pi_i$ .

Let  $V_n[x, \Pi]$  be the optimal expected utility for an  $n$  stage game when our initial fortune is  $x$  and we have a prior probability distribution  $\Pi$ .

Theorem 5.8:

$V_n[x, \Pi]$  is concave in  $\Pi$ .

Proof:

Let  $V_n[x, \Pi, \delta(x)]$  be the expected utility in an  $n$  stage game when our fortune is  $x$ , our prior distribution is  $\Pi$  and we use a betting strategy  $\delta(x)$ .

Clearly

$$V_n[x, \Pi] = \sup_{\delta(x)} \{V_n[x, \Pi, \delta(x)]\}.$$

Let  $B_n[x, p, \delta(x)]$  be the expected utility in an  $n$  stage game when our initial fortune is  $x$ , the probability of winning is  $p$  and we are using strategy  $\delta(x)$ .

Clearly

$$V_n[x, \Pi, \delta(x)] = \sum_i \pi_i B_n[x, p_i, \delta(x)].$$

Now consider two prior distributions  $\Pi^1$  and  $\Pi^2$  and any real numbers  $\lambda \in [0, 1]$ .

$$\begin{aligned} V_n[x, \lambda \Pi^1 + (1 - \lambda) \Pi^2] &= \\ \sup_{\delta(x)} \{V_n[x, \lambda \Pi^1 + (1 - \lambda) \Pi^2, \delta(x)]\} &= \\ \sup_{\delta(x)} \left\{ \sum_i \lambda \pi_i^1 + (1 - \lambda) \pi_i^2 B_n[x, p_i, \delta(x)] \right\} &\leq \\ \lambda \sup_{\delta(x)} \left\{ \sum_i \pi_i^1 B_n[x, p_i, \delta(x)] \right\} + (1 - \lambda) \sup_{\delta(x)} \left\{ \sum_i \pi_i^2 B_n[x, p_i, \delta(x)] \right\} &\leq \\ \lambda V_n[x, \Pi^1] + (1 - \lambda) V_n[x, \Pi^2]. \end{aligned}$$

Thus proving the convexity of  $V_n[x, \Pi]$  in  $\Pi$ . ■

In order to analyze monotonicity properties of  $V_n[x, \Pi]$  in  $\Pi$  it is necessary to impose some ordering on the collection of allowable  $\Pi$ 's.

We will assume that the possible values of the probability of winning,  $p_i$ , are ordered so that  $p_1 < p_2 < \dots$ .

Let  $\Pi = \{\pi_1, \pi_2, \dots\}$  and  $\Omega = \{w_1, w_2, \dots\}$  be two prior densities. We will say  $\Pi$  (strictly) dominates  $\Omega$  ( $\Pi \succeq \Omega$  or  $\Pi \succ \Omega$ ) if and only if for all  $i$  and  $j$   $i < j$  such that both mass functions do not vanish identically we have  $\frac{\pi_i}{w_i} \leq \frac{\pi_j}{w_j}$  (or with strict inequality).

This definition is the usual Monotone Likelihood Ratio.

It is well known that if  $\Pi$  (strictly) dominates  $\Omega$  then  $\Pi$  is stochastically larger (strictly) than  $\Omega$ . Hence it follows

$$\sum_i p_i \pi_i \geq \sum_i p_i w_i$$

(or with strict inequality).

If  $\Pi$  is a prior density, we will call  $W(\Pi)$  and  $L(\Pi)$  the posterior densities given we have won a game and that we have lost one respectively.

Clearly

$$W(\Pi) = \frac{\left\{ p_i \pi_i \right\}}{\left\{ \sum_j p_j \pi_j \right\}}$$

$$L(\Pi) = \frac{\left\{ (1 - p_i) \pi_i \right\}}{\left\{ 1 - \sum_j p_j \pi_j \right\}}.$$



We will need the following.

Proposition 5.8:

Let  $\Pi$  and  $\Omega$  be two prior densities. Assume  $\Pi \succeq \Omega$  then.

a)  $W(\Pi) \succeq \Pi \succeq L(\Pi)$

b)  $W(\Pi) \succeq W(\Omega)$

c)  $L(\Pi) \succeq L(\Omega)$  .

If  $\Pi \succ \Omega$  then b and c are strict.

Proof:

To show a) we form the ratios

$$\frac{1}{\pi_i} \frac{p_i \pi_i}{\sum_j p_j \pi_j} = \frac{p_i}{\sum_j p_j \pi_j}$$

and

$$\pi_i / \left( \frac{(1 - p_i) \pi_i}{1 - \sum_j p_j \pi_j} \right) = \frac{1 - \sum_j p_j \pi_j}{1 - p_i}$$

which are clearly increasing in  $i$  since  $p_i$  is .

To show b) and c) we form the ratios

$$\frac{p_i \pi_i}{\sum_j p_j \pi_j} / \frac{p_i w_i}{\sum_j p_j w_j} = \frac{\sum_j p_j w_j}{\sum_j p_j \pi_j} \frac{\pi_i}{w_i}$$

and

$$\frac{(1 - p_i)\pi_i}{1 - \sum p_j \pi_j} \bigg/ \frac{(1 - p_i)w_i}{1 - \sum p_j w_j} = \frac{1 - \sum p_j w_j}{1 - \sum p_j \pi_j} \frac{\pi_i}{w_i}$$

which are clearly increasing in  $i$  since  $\frac{\pi_i}{w_i}$  is . The proof in the strict case is of course identical. ■

We are now in a position to prove that the ordering of the prior densities defined above induces an ordering of the optimal value function.

Theorem 5.9:

Let  $\Pi$  and  $\Omega$  be two prior densities. If  $\Pi \succeq \Omega$ , then for all  $n$  and  $x$   $V_n[x, \Pi] \geq V_n[x, \Omega]$ .

Proof:

The proof proceeds by induction on  $n$ . For  $n = 0$   $V_0[x, \Pi] = V[x] = V_0[x, \Omega]$  for all  $x$ ,  $\Pi$  and  $\Omega$ . Assume  $V_{n-1}[x, \Pi] \geq V_{n-1}[x, \Omega]$  for all  $x$ ,  $\Pi$  and  $\Omega$  such that  $\Pi \succeq \Omega$ .

$$V_n[x, \Pi] = \max_{0 \leq y \leq x} \left\{ \left( \sum p_i \pi_i \right) V_{n-1}[x + y, W(\Pi)] + \left( 1 - \sum p_i \pi_i \right) V_{n-1}[x - y, L(\Pi)] \right\}.$$

By Proposition 5.8  $W(\Pi) \succeq W(\Omega)$  and  $L(\Pi) \succeq L(\Omega)$ . Hence by the induction hypothesis

$$\begin{aligned}
V_n[x, \Pi] &\geq \max_{0 \leq y \leq x} \left\{ \left( \sum p_i \pi_i \right) V_{n-1}[x+y, W(\Omega)] + \right. \\
&\quad \left. \left( 1 - \sum p_i \pi_i \right) V_{n-1}[x-y, L(\Omega)] \right\} \\
&\geq \max_{0 \leq y \leq x} \left\{ \left( \sum p_i \pi_i \right) (V_{n-1}[x+y, W(\Omega)] - \right. \\
&\quad \left. V_{n-1}[x-y, L(\Omega)]) + V_{n-1}[x-y, L(\Omega)] \right\}.
\end{aligned}$$

By Theorem 5.2  $V_{n-1}[x, \Pi]$  is increasing in  $x$ . Thus,

$$V_{n-1}[x+y, W(\Omega)] - V_{n-1}[x-y, L(\Omega)] \geq V_{n-1}[x-y, W(\Omega)] - V_{n-1}[x-y, L(\Omega)]$$

and, since, by Proposition 5.8  $W(\Omega) \succeq L(\Omega)$ , we have by the induction hypothesis

$$V_{n-1}[x+y, W(\Omega)] - V_{n-1}[x-y, L(\Omega)] \geq 0.$$

Thus, since  $\sum p_i \pi_i \geq \sum p_i w_i$ , we have

$$\begin{aligned}
V_n[x, \Pi] &\geq \max_{0 \leq y \leq x} \left\{ \left( \sum p_i w_i \right) (V_{n-1}[x+y, W(\Omega)] - V_{n-1}[x-y, L(\Omega)]) + \right. \\
&\quad \left. V_{n-1}[x-y, L(\Omega)] \right\}
\end{aligned}$$

which is

$$V_n[x, \Pi] \geq V_n[x, \Omega]$$

completing the induction. ■

The question now arises as to the dependence of the optimal wager  $y_n(x, \Pi)$  on the density  $\Pi$ . One would wish to show that if  $\Pi \succeq \Omega$  then  $y_n(x, \Pi) \geq y_n(x, \Omega)$ .



The proof of this monotonicity requires however, as in the non-adaptive case, an indirect approach, based on examining games in which different gambles are available and showing the irrelevance of the order in which the gambles are chosen.

Suppose a player faces  $n$  coin tosses, with different coins. He can choose the order in which he plays the coins but each coin can only be used once. His gambling universe is in some state  $i \in \{1, 2, \dots\}$  unknown except for a prior distribution. Let  $\pi_i = P[\text{state of the universe is } i]$ . If the state of the universe is  $i$ , and he chooses coin  $j$ , he will win with probability  $p_{ji}$  and lose with probability  $q_{ji} = 1 - p_{ji}$ . The state of the universe remains fixed throughout the game, but is unknown (except for the prior distribution) to the gambler, who gains information solely on the basis of the outcome of his bets. As usual the gambler acts so as to maximize the expected value of his utility function  $V[\cdot]$ .

Let  $V_n[x, \Pi]$  be the expected utility in an  $n$  stage game if one starts with a fortune of  $x$  and a prior distribution  $\Pi$  and proceeds optimally. Let  $y_n[x, \Pi]$  be the optimal wager in this case. Note the dependence of these two functions on the values and order of the  $p_{ji}$  which has been suppressed for simplicity of notation. We will show that, under some conditions, the order in which the coins are chosen is irrelevant.

Theorem 5.10:

In the game described above, assume  $p_{ji} > \frac{1}{2}$  for all  $i$  and  $j$ . Further assume  $V[\cdot]$  is strictly increasing, differentiable and strictly concave.

Then  $V_n[x, \Pi]$  is independent of the order in which the coins are chosen, and for all possible orders  $y_n(x, \Pi) > 0$  for all  $x$  and  $\Pi$ .

Proof:

The proof proceeds by induction on  $n$ .

$V_1[x, \Pi]$  clearly does not depend on the order of the coins, since there is only one coin.  $y_1(x, \Pi)$  is chosen by solving

$$\max_{0 \leq y \leq x} \left\{ \left( \sum p_{1i} \pi_i \right) V[x + y] + \left( \sum q_{1i} \pi_i \right) V[x - y] \right\}$$

by the differentiability of  $V$  we obtain at  $y = 0$

$$\sum (p_{1i} - q_{1i}) \pi_i V'[x] > 0$$

since  $V$  is strictly increasing and  $p_{1i} > q_{1i}$  for all  $i$ .

By the strict concavity and smoothness properties of  $V$  this implies  $y_1(x, \Pi) > 0$ .

Now assume that in any  $n - 1$  stage  $V_{n-1}[x, \Pi]$  does not depend on the order of the coins, and for any order chosen  $y_{n-1}(x, \Pi) > 0$ .

Now consider an  $n$  stage game where coins  $1, 2, \dots, n$  are available. Assume we pick coin 1 to be played first. Whatever the outcome of the game, we will be faced with an  $n - 1$  coin game with coins  $2, 3, \dots, n$ . By the induction hypothesis any of these can be chosen to continue. Assume the gambler chooses coin 2 whether or not he wins on the first toss. We will prove that the gambler could have done just as well by choosing coin 2 as the first

coin. If the gambler chooses coin 1 and then coin 2, his optimal expected utility is given by

$$\begin{aligned} \max_{y, y_W, y_L} \left\{ & V_{n-2}[x + y + y_W, W_2 W_1(\Pi)] \sum p_{1i} p_{2i} \pi_i + \right. \\ & V_{n-2}[x + y - y_W, L_2 W_1(\Pi)] \sum p_{1i} q_{2i} \pi_i + \\ & V_{n-2}[x - y + y_L, W_2 L_1(\Pi)] \sum q_{1i} p_{2i} \pi_i + \\ & \left. V_{n-2}[x - y - y_L, L_2 L_1(\Pi)] \sum q_{1i} q_{2i} \pi_i \right\} \\ \text{subject to } & y \geq 0 \quad x \geq y \\ & y_W \geq 0 \quad x + y \geq y_W \\ & y_L \geq 0 \quad x - y \geq y_L \end{aligned}$$

where  $y$  is the optimal wager in the first stage,  $y_W$  and  $y_L$  are the optimal wagers in the second stage if one wins or loses respectively in the first stage.  $W_1(\Pi)$  and  $L_1(\Pi)$  represent the posterior distribution if one respectively wins or loses in a loss of coin 1 and the prior distribution is  $\Pi$ .

Clearly

$$W_2 W_1(\Pi) = W_1 W_2(\Pi)$$

$$L_2 W_1(\Pi) = W_1 L_2(\Pi)$$

$$W_2 L_1(\Pi) = L_1 W_2(\Pi)$$

$$L_2 L_1(\Pi) = L_1 L_2(\Pi) .$$

Let  $y^*$ ,  $y_W^*$ ,  $y_L^*$  be the optimal values of the variables in this optimization problem. They are unique since we are maximizing



a strictly concave function over a compact set. Clearly by the induction hypothesis

$$\begin{aligned}
 y^* &= y_n(x, \Pi) \\
 (5.1) \quad y_W^* &= y_{n-1}[x + y^*, W_1(\Pi)] > 0 \\
 y_L^* &= y_{n-1}[x - y^*, L_1(\Pi)] > 0.
 \end{aligned}$$

Since we know that in the optimal solution  $y_W^* > 0$  and  $y_L^* > 0$ , the nonnegativity constraints on  $y_W$  and  $y_L$  can be relaxed because of the strict concavity of the objective function and the convexity of the constraint set. However we need to guarantee the nonnegativity of the first arguments of  $v_{n-2}$ , and hence the nonnegativity constraints may be replaced by

$$x + y + y_W \geq 0 \quad x - y + y_L \geq 0.$$

Now consider playing coin 2 first, followed by coin 1.

The optimal strategy is given by

$$\begin{aligned}
 \max_{y, y_W, y_L} \quad & \left\{ v_{n-2}[x + y + y_W, W_1 W_2(\Pi)] \sum p_{2i} p_{1i} \pi_i + \right. \\
 & v_{n-2}[x + y - y_W, L_1 W_2(\Pi)] \sum p_{2i} q_{2i} \pi_i + \\
 & v_{n-2}[x - y + y_L, W_1 L_2(\Pi)] \sum q_{2i} p_{1i} \pi_i + \\
 & \left. v_{n-2}[x - y - y_L, L_1 L_2(\Pi)] \sum q_{2i} q_{1i} \pi_i \right\} \\
 \text{subject to} \quad & y \geq 0 \quad x + y \geq y_W \quad x + y + y_W \geq 0 \\
 & x \geq y \quad x - y \geq y_L \quad x - y + y_L \geq 0.
 \end{aligned}$$

Since again the nonnegativity constraints on  $y_W$  and  $y_L$  may be relaxed.

Define

$$\begin{aligned}
 y^+ &= \frac{1}{2} (y_W^* + y_L^*) \\
 (5.2) \quad y_W^+ &= y^* + \frac{1}{2} (y_W^* - y_L^*) \\
 y_L^+ &= y^* - \frac{1}{2} (y_W^* - y_L^*) .
 \end{aligned}$$

We now proceed to verify that these satisfy the constraints of the above maximization problem

$$\begin{aligned}
 y_W^* > 0 \text{ and } y_L^* > 0 &\text{ imply } y^+ > 0 \\
 x + y^* \geq y_W^* \text{ and } x - y^* \geq y_L^* &\text{ imply } x \geq \frac{1}{2} (y_W^* + y_L^*) = y^+ \\
 x + y^+ - y_W^+ = x - y^* + y_L^* \geq 0 ; &x + y^+ + y_W^+ = x + y^* + y_W^* \geq 0 \\
 x - y^+ - y_L^+ = x - y^* - y_L^* \geq 0 ; &x - y^+ + y_L^+ = x + y^* - y_W^* \geq 0 .
 \end{aligned}$$

Thus  $y^+$ ,  $y_W^+$ ,  $y_L^+$  are feasible for this second maximization problem. Direct substitution in the objective function shows that they yield the same value of the objective function as was optimal for the first problem. Thus playing coin 2 first and then coin 1 is at least as good as playing coin 1 and then coin 2. Since the numbering of the coins was arbitrary, reversal of the argument shows that playing coin 1 first is at least as good as playing coin 1 first. Hence both alternatives are equivalent.

Also,  $y^+$ ,  $y_W^+$ ,  $y_L^+$  are optimal for the second problem. However, by the strict concavity of the objective function in this problem, these must be the unique solution to the maximization problem. Thus, since

$$y^+ = \frac{1}{2} (y_W^* + y_L^*) > 0$$

$$y^* = \frac{1}{2} (y_W^+ + y_L^+) > 0$$

we have that  $y_n(x, \Pi) > 0$  completing the induction. ■

The question of whether one can drop the assumption that  $p_{ji} > \frac{1}{2}$  is answered in the negative in the following counter example.

There are two states of the universe and two coins. Coin 1 has  $p_{11} = 1$ ,  $p_{12} = 0$ . Coin 2 has  $p_{21} = 0$ ,  $p_{22} = 1$ . The prior distribution is  $\pi_1 = \frac{3}{4}$ ,  $\pi_2 = \frac{1}{4}$  and the utility function is  $\text{Ln} x$ .

Choosing coin 1 first and coin 2 second

$$V_2[x, \Pi] = \text{Ln} x + .3041, \quad y_2(x, \Pi) = \frac{1}{2} x.$$

Choosing coin 2 first and coin 1 second

$$V_2[x, \Pi] = \text{Ln} x + .5199, \quad y_2(x, \Pi) = 0.$$

Thus one should choose coin 2 first and then coin 1. The reason for the failure is of course that with some  $p_{ji} < \frac{1}{2}$  we find it desirable to bet 0. Hence  $y_n(x, \Pi) = 0$  and the non-negativity constraint may not be dropped as we did in our proof.



Now consider a gambler who faces  $n$  tosses of the same coin. He does not know the probability of winning, except that it is one of  $p_1 < p_2 < p_3 < \dots$ . He knows a prior density  $\Pi = [\pi_1, \pi_2, \dots]$  with the interpretation  $\pi_i = P[\text{the coin is of type } i]$ . We wish to study the dependence of his optimal wager  $y_n(x, \Pi)$  on  $\Pi$ .

We first give a proposition, useful in its own right, that gives a recurrence relation satisfied by  $y_n(x, \Pi)$ .

Proposition 5.11:

Assume  $p_i > \frac{1}{2}$  for all  $i$ . Further assume  $V[\cdot]$  is strictly increasing, differentiable and strictly concave. Then

$$y_n(x, \Pi) > 0$$

and

$$y_n(x, \Pi) = \frac{1}{2} \{y_{n-1}[x + y_n(x, \Pi), W(\Pi)] + y_{n-1}[x - y_n(x, \Pi), L(\Pi)]\}.$$

Proof:

In the game of Theorem 5.10, set  $p_{ji} = p_i$  for  $j = 1, 2, \dots, n$ . Then, exchanging coins 1 and 2 does not change the optimal policy. Hence  $y^+ = y^*$ . Also Equation (5.2) apply. These, together with (5.1) imply the desired results. ■

Theorem 5.12:

Let  $\Pi$  and  $\Omega$  be two prior distributions. Assume  $p_i > \frac{1}{2}$  for all  $i$ . Further, assume  $V[\cdot]$  is strictly increasing, differentiable and strictly concave. If  $\Pi$  strictly dominates  $\Omega$ ,

then for all  $n$  and  $x$   $y_n(x, \Pi) \geq y_n(x, \Omega)$ . Furthermore if  $y_n(x, \Omega) < x$ , then  $y_n(x, \Pi) > y_n(x, \Omega)$ .

Proof:

Our proof proceeds by induction. For  $n = 1$ ,  $y_1(x, \Omega)$  is given by

$$\max_{0 \leq y \leq x} \left\{ V[x + y] \sum p_i w_i + V[x - y] \left( 1 - \sum p_i w_i \right) \right\}.$$

If  $y_1(x, \Omega) = x$ , then we must have

$$V'[2x] \sum p_i w_i \geq V'[0] \left( 1 - \sum p_i w_i \right).$$

Since  $\sum p_i \pi_i > \sum p_i w_i$  and  $V$  is strictly increasing

$$V'[2x] \sum p_i \pi_i > V'[2x] \sum p_i w_i \geq V'[0] \left( 1 - \sum p_i w_i \right) > V'[0] \left( 1 - \sum p_i \pi_i \right)$$

implying  $y_1(x, \Pi) = x$ .

If  $0 < y_1(x, \Omega) < x$  then we must have

$$V'[x + y_1(x, \Omega)] \sum p_i w_i = V'[x - y_1(x, \Omega)] \left( 1 - \sum p_i w_i \right).$$

The same argument shows thus that

$$V'[x + y_1(x, \Omega)] \sum p_i \pi_i - V'[x - y_1(x, \Omega)] \left( 1 - \sum p_i \pi_i \right) > 0$$

implying  $y_1(x, \Pi) > y_1(x, \Omega)$ .

By Proposition 5.11, the case  $y_n(x, \Omega) = 0$  does not arise.

To simplify notation, let  $y^+ = y_n(x, \Omega)$ ,  $y^* = y_n(x, \Pi)$ .

Assume the theorem is true for  $n - 1$ . Then by Proposition 5.11

$$(5.3) \quad \begin{aligned} y^+ &= \frac{1}{2} \left\{ y_{n-1}[x + y^+, W(\Omega)] + y_{n-1}[x - y^+, L(\Omega)] \right\} \\ y^* &= \frac{1}{2} \left\{ y_{n-1}[x + y^*, W(\Pi)] + y_{n-1}[x - y^*, L(\Pi)] \right\}. \end{aligned}$$

Then

$$(5.4) \quad \begin{aligned} y^+ - y^* &= \frac{1}{2} \left\{ y_{n-1}[x + y^+, W(\Omega)] + y_{n-1}[x - y^+, L(\Omega)] - \right. \\ &\quad \left. y_{n-1}[x + y^*, W(\Pi)] - y_{n-1}[x - y^*, L(\Pi)] \right\}. \end{aligned}$$

If  $y^+ < x$  then by 5.3, either

$$y_{n-1}[x + y^+, W(\Omega)] < x + y^+$$

or

$$y_{n-1}[x - y^+, L(\Omega)] < x - y^+$$

or both.

By the induction hypothesis, in the first case

$$y_{n-1}[x + y^+, W(\Omega)] < y_{n-1}[x + y^+, W(\Pi)]$$

and in the second case

$$y_{n-1}[x - y^+, L(\Omega)] < y_{n-1}[x - y^+, L(\Pi)].$$

In either case 5.4 can be written as



$$y^+ - y^* < \frac{1}{2} \left\{ y_{n-1}[x + y^+, W(\Pi)] - y_{n-1}[x + y^*, W(\Pi)] + \right. \\ \left. y_{n-1}[x - y^+, L(\Pi)] - y_{n-1}[x - y^*, L(\Pi)] \right\}.$$

By Corollary 5.7

$$y_{n-1}[x + y^+, W(\Pi)] - y_{n-1}[x + y^*, W(\Pi)] \leq |y^+ - y^*| \\ y_{n-1}[x - y^+, L(\Pi)] - y_{n-1}[x - y^*, L(\Pi)] \leq |y^+ - y^*|.$$

Hence

$$y^+ - y^* < |y^+ - y^*| \text{ implying } y^+ - y^* < 0 \text{ that is.}$$

$$y_n(x, \Pi) > y_n(x, \Omega) \text{ if } y_n(x, \Omega) < x.$$

If  $y^+ = x$  we must show  $y^* = x$ . Assume  $y^* < x$ . Then, by 5.3, either

$$y_{n-1}[x + y^*, W(\Pi)] < x + y^*$$

or

$$y_{n-1}[x - y^*, L(\Pi)] < x - y^*,$$

or both.

In the first case

$$y_{n-1}[x + y^*, W(\Omega)] < y_{n-1}[x + y^*, W(\Pi)]$$

and in the second case

$$y_{n-1}[x - y^*, L(\Omega)] < y_{n-1}[x - y^*, L(\Pi)] .$$

In either case 5.4 can be rewritten as

$$y^+ - y^* < \frac{1}{2} \left\{ y_{n-1}[x + y^+, W(\Omega)] - y_{n-1}[x + y^*, W(\Omega)] + \right. \\ \left. y_{n-1}[x - y^+, L(\Omega)] - y_{n-1}[x - y^*, L(\Omega)] \right\}$$

and the same argument shows that  $y^+ < y^*$ . Since we had assumed  $y^+ = x$ ,  $y^* < x$ , this is a contradiction. Hence we must have  $y^* = x$ . This shows that  $y_n(x, \Omega) = x$  implies  $y_n(x, \Pi) = x$  completing the induction. ■

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## APPENDIX

## SUPERADDITIVE FUNCTIONS AND ISOTONICITY THEOREM

The results in this appendix are not original. They were taught to the author, during a course in inventory theory, by Donald Topkis, and are included here for the sake of completeness, since no reference could be found in an exhaustive literature search.

We shall be concerned with the following problem. Let  $f(x,y) : S \rightarrow R$  where  $x \in E^n$   $y \in E^m$ ,  $S \subset E^n \times E^m$ .

Consider the problem  $\max_x f(x,y)$  where the maximization is carried out over  $x : (x,y) \in S$ . Let  $x(y)$  be the vector  $x$  where the maximum is attained.

When, and how, can we say  $x(y)$  is in some sense increasing in  $y$ ?

For any real numbers  $x$  and  $y$  we define the operators  $\vee$  and  $\wedge$  as follows

$$x \vee y = \max \{x, y\} \quad x \wedge y = \min \{x, y\}.$$

The definition can be extended to  $n$ -vectors in the obvious way: for  $x, y \in E^n$   $x \vee y = (x_1 \vee y_1, x_2 \vee y_2, \dots, x_n \vee y_n)$  and similarly for  $\wedge$ .

Definition A1:

A set  $S \subset E^n$  is a sublattice if  $\forall x, y \in S$   $x \vee y \in S$  and  $x \wedge y \in S$ .

Examples of sublattices follow.

A1.1.  $E^n$ , the complete space

A1.2.  $\emptyset$ , the null set

A1.3. any subset of  $E^1$

A1.4.  $\{x \in E^n : x \leq b\}$

A1.5.  $Z^n$ , the set of all n-tuples of rationals

A1.6.  $\{x \in E^n : \lambda x_i - \mu x_j \leq \alpha\}$  if  $\lambda, \mu \geq 0$ .

The proofs that these are sublattices are trivial.

Lemma A1:

If  $S_\gamma$  is a sublattice  $\forall \gamma \in \Gamma$ , then so is  $\bigcap_{\gamma} S_\gamma$ .

Proof:

Elementary. ■

If  $S$  is a sublattice of  $E^n \times E^m$ , then for each  $y \in E^m$  define

$S_y = \{x \in E^n : (x, y) \in S\}$  the section of  $S$  at  $y$

$\Pi_y S = \{y \in E^m : S_y \neq \emptyset\}$  the projection of  $S$  on the  $y$  axis.

Lemma A2:

$S_y$  and  $\Pi_y S$  are sublattices.

Proof:

Easy. ■



Definition A2:

A function  $f : S \rightarrow R$  is superadditive on the sublattice  $S \subset E^n$  if for all  $x^1, x^2 \in S$

$$f(x^1 \vee x^2) + f(x^1 \wedge x^2) \geq f(x^1) + f(x^2) .$$

Theorem A3:

$f : S \rightarrow R$  is superadditive on the sublattice  $S \subset E^n$  if and only if for all  $i$   $f(x + \epsilon \mu_i) - f(x)$  is increasing in  $x_j$   $j \neq i$ , where  $\mu_i$  represents the  $i$ -th unit vector.

Proof:

Suppose  $f$  is superadditive. We have to show, for  $\epsilon, \delta > 0$

$$f(x + \epsilon \mu_i) - f(x) \leq f(x + \epsilon \mu_i + \delta \mu_j) - f(x + \delta \mu_j) \quad j \neq i .$$

It is clear that

$$(x + \epsilon \mu_i) \wedge (x + \delta \mu_j) = x, \quad (x + \epsilon \mu_i) \vee (x + \delta \mu_j) = x + \epsilon \mu_i + \delta \mu_j .$$

By superadditivity

$$\begin{aligned} f(x + \epsilon \mu_i) + f(x + \delta \mu_j) &\leq \\ f((x + \epsilon \mu_i) \wedge (x + \delta \mu_j)) + f((x + \epsilon \mu_i) \vee (x + \delta \mu_j)) & \\ f(x + \epsilon \mu_i) + f(x + \delta \mu_j) &\leq f(x) + f(x + \epsilon \mu_i + \delta \mu_j) \end{aligned}$$

which establishes the necessity.

Conversely, pick  $x, y \in S$ . If  $x = y$  then

$$f(x \vee y) + f(x \wedge y) = f(x) + f(y).$$

If there exists an  $i$  such that  $x_i \neq y_i$ ,  $x_j = y_j \quad \forall j \neq i$  then

$$f(x \vee y) + f(x \wedge y) = f(x) + f(y).$$

Now assume there are two or more indices  $i$  such that  $x_i \neq y_i$ .

Assume we have relabeled the indices so that  $x_1 \neq y_1$ ,

$$x_2 \neq y_2 \cdots x_p \neq y_p, x_{p+1} = y_{p+1} \cdots x_n = y_n.$$

Consider  $p = 2$ .

There are four cases

$$a) \quad x_1 \leq y_1 \quad x_2 \leq y_2$$

Then  $x \wedge y = x$   $x \vee y = y$  and  $f(x \wedge y) + f(x \vee y) = f(x) + f(y)$ .

$$b) \quad x_1 \geq y_1 \quad x_2 \geq y_2. \quad \text{A similar argument hold}$$

$$c) \quad x_1 \geq y_1 \quad x_2 \leq y_2$$

Then  $x_1 = y_1 + \varepsilon$   $y_2 = x_2 + \delta$ . The condition states

$$f(y + \varepsilon \mu_1) - f(y) \geq f(y + \varepsilon \mu_1 - \delta \mu_2) - f(y - \delta \mu_2)$$

$$f(x \vee y) - f(y) \geq f(x) - f(x \wedge y).$$

Since

$$x = y + \varepsilon \mu_1 - \delta \mu_2$$

$$x \wedge y = y - \delta \mu_2$$

$$x \vee y = y + \varepsilon \mu_1.$$

d)  $x_1 \leq y_1$      $x_2 \geq y_2$  . A similar proof holds.

For  $p > 2$  we apply repeatedly the same argument, increasing step by step along each coordinate. ■

#### Examples of superadditive functions

A2.1. Any real function on  $E^1$

A2.2. The sum of superadditive functions

A2.3. A positive constant times a superadditive function

A2.4. A separable function  $f(x) = \sum_{i=1}^n f_i(x_i)$

A2.5. If  $f$  is concave on  $E^1$  then  $g(x_1, x_2) = f(x_1 - x_2)$

is superadditive in the pair because

$$g(x_1 + \epsilon, x_2) - g(x_1, x_2) = f(x_1 + \epsilon - x_2) - f(x_1 - x_2)$$

increasing in  $x_2$  by the concavity of  $f$  . The

converse is also true.

The proof of these examples is trivial.

#### Lemma A4:

Let  $f$  be superadditive on a sublattice  $S \subset E^n$  and let  $S^* = \{x \in S : f(x) \geq f(y) \ \forall y \in S\}$  . Then  $S^*$  is a sublattice.

#### Proof:

Pick  $x^1, x^2 \in S^*$  . Since  $S^* \subset S$      $x^1 \vee x^2 \in S$      $x^1 \wedge x^2 \in S$  .

By superadditivity of  $f$  and optimality of  $x^1$  and  $x^2$

$$0 \leq f(x^1 \vee x^2) - f(x^1) \leq f(x^2) - f(x^1 \wedge x^2) \leq 0 .$$



Thus  $f(x^1 \vee x^2) = f(x^1) = f(x^2) = f(x^1 \wedge x^2)$  the maximal value of  $f$ , implying  $x^1 \vee x^2 \in S^*$ ,  $x^1 \wedge x^2 \in S^*$ . ■

Theorem A5:

Assume  $S$  is a nonempty compact sublattice of  $E^n$ . Then  $S$  has a greatest element  $\bar{x}$  and a smallest element  $\underline{x}$ .

Proof:

We have to show that there exist  $\bar{x}$  and  $\underline{x}$  such that

$$\forall x \in S \quad \bar{x} \vee x = \bar{x} \quad \underline{x} \wedge x = \underline{x}.$$

For  $i = 1, 2, \dots, n$  pick  $\tilde{x}^i \in E^n$  to be the solution to

$$\max x_i$$

subject to  $x \in S$ .

$\tilde{x}^i$  exist since we are maximizing a continuous function over a nonempty compact set  $S$ .

Pick  $\bar{x} = \bigvee_{i=1}^n \tilde{x}^i$ . Clearly  $\bar{x} \in S$  since  $\tilde{x}^i \in S \forall i$ .  
 $\bar{x}_i \geq \tilde{x}_i^1 \geq x_i \quad \forall x \in S, \forall i$  so  $\bar{x} \vee x = \bar{x} \quad \forall x \in S$ . A similar construction with minimization instead of maximization yields  $\underline{x}$ . ■

Corollary A6:

If the superadditive function  $f$  is continuous and the set  $S$  is compact then the set  $S^*$  of Lemma A4 is nonempty and hence has a greatest and least element.

Theorem A7:

Let  $S \subset E^n \times E^m$  be a sublattice. For  $x \in E^n$ ,  $y \in E^m$ .

Assume  $f(x,y) : S \rightarrow R$  is superadditive on  $S$ . Then

$$g(y) = \max_{x \in S_y} f(x,y)$$

is superadditive on  $\Pi_y S$ .

Proof:

Pick  $y^1, y^2 \in \Pi_y S$ . Let  $x^1$  and  $x^2$  be optimal for  $y^1$  and  $y^2$  respectively. Then

$$\begin{aligned} g(y^1) + g(y^2) &= f(x^1, y^1) + f(x^2, y^2) \\ &\leq f(x^1 \wedge x^2, y^1 \wedge y^2) + f(x^1 \vee x^2, y^1 \vee y^2) \\ &\leq g(y^1 \wedge y^2) + g(y^1 \vee y^2) . \blacksquare \end{aligned}$$

Definition A3:

Let  $A$  and  $B$  be sublattices of  $E^n$ . We say  $B$  dominates  $A$  ( $A \preceq B$ ) if whenever  $a \in A$ ,  $b \in B$  we have  $a \wedge b \in A$ ,  $a \vee b \in B$ .

Lemma A8:

The set of all nonempty sublattices of  $E^n$  is partially ordered by the relation  $\preceq$ .

Proof:

We need to show

- a)  $A \preceq A$  trivially true
- b)  $A \preceq B$  and  $B \preceq A$  implies  $A = B$ .

Choose  $b \in B$ . Then for  $a \in A$   $a \wedge b \in A$  because  $A \preceq B$ .

But  $b \in B$  thus  $b \vee (a \wedge b) \in A$  because  $B \preceq A$ . But

$b \vee (a \wedge b) = b$  so  $b \in A$ .

- c)  $A \preceq B$   $B \preceq C$  implies  $A \preceq C$ .

By enumeration of cases one can verify that for  $a \in A$ ,  $b \in B$

$c \in C$   $a \vee c = [(b \wedge c) \vee a] \vee c$ . Then  $b \wedge c \in B$  ( $B \preceq C$ ) so

$$(b \wedge c) \vee a \in B \quad (A \preceq B)$$

so

$$a \vee c = [(b \vee c) \vee a] \vee c \in C \quad (B \preceq C).$$

Similarly by noting that  $a \wedge c = [(a \vee b) \wedge c] \wedge a$  we can show

$a \wedge c \in A$ . ■

Definition A4:

If  $T \subset E^m$  is partially ordered by some relation  $\subset$  we say  $S_t \subset E^n$  is ascending in  $t$  on  $T$  if  $t \subset t'$  implies  $S_t \preceq S_{t'}$ .



Lemma A9:

If  $S$  is a sublattice of  $E^n \times E^m$  then  $S_y$  is ascending in  $y \in E^m$  on  $\Pi_y S$  under the usual vector ordering.

Proof:

Pick  $y^1, y^2 \in \Pi_y S$   $y^1 \leq y^2$ .

Pick  $x^1 \in S_{y^1}$   $x^2 \in S_{y^2}$ . Then  $(x^1, y^1) \in S$ ,  $(x^2, y^2) \in S$

so

$$(x^1 \wedge x^2, y^1 \wedge y^2) \in S \Rightarrow (x^1 \wedge x^2, y^1) \in S \Rightarrow x^1 \wedge x^2 \in S_{y^1}$$

$$(x^1 \vee x^2, y^1 \vee y^2) \in S \Rightarrow (x^1 \vee x^2, y^2) \in S \Rightarrow x^1 \vee x^2 \in S_{y^2}. \blacksquare$$

Lemma A10:

If  $A$  and  $B$  are sublattices of  $E^n$  with  $A \leq B$ , and both  $A$  and  $B$  have greatest (least) elements  $\bar{a}$ ,  $\bar{b}$  ( $\underline{a}, \underline{b}$ ) respectively, then  $\bar{a} \leq \bar{b}$  ( $\underline{a} \leq \underline{b}$ ).

Proof:

Since  $A \leq B$  we have  $\bar{a} \vee \bar{b} \in B$  implying  $\bar{a} \vee \bar{b} \leq \bar{b}$  since  $\bar{b}$  is the greatest element of  $B$ . But  $\bar{b} \leq \bar{a} \vee \bar{b}$  by the definition of the  $\vee$  operator. Thus  $\bar{a} \leq \bar{a} \vee \bar{b} = \bar{b}$ . Similarly  $\underline{a} \wedge \underline{b} \in A \Rightarrow \underline{a} \wedge \underline{b} \geq \underline{a}$  but  $\underline{a} \wedge \underline{b} \leq \underline{a}$  so  $\underline{a} = \underline{a} \wedge \underline{b} \leq \underline{b}$ .  $\blacksquare$

We now come to the main result in this appendix.

Theorem All: Isotonicity Theorem

Let  $f$  be superadditive on a sublattice  $S \subset E^n \times E^m$ .  
 For  $x \in E^n$ ,  $y \in E^m$  let  $S_y^*$  be the set of all optimal  $x$  for  
 the problem  $\max \{f(x, y) : x \in S_y\}$ .

Then  $S_y^*$  is ascending in  $y$  on  $y : S_y^* \neq \emptyset$ .

Proof:

Pick  $y^1$  and  $y^2$  such that  $y^1 \leq y^2$ ,  $S_{y^1}^* \neq \emptyset$ ,  $S_{y^2}^* \neq \emptyset$ .  
 We will show  $S_{y^1}^* \preceq S_{y^2}^*$ .  
 Choose  $x^1 \in S_{y^1}^*$ ,  $x^2 \in S_{y^2}^*$ . Clearly  $(x^1, y^1) \in S$   
 $(x^2, y^2) \in S$ . Also

$$(x^1, y^1) \wedge (x^2, y^2) = (x^1 \wedge x^2, y^1)$$

$$(x^1, y^1) \vee (x^2, y^2) = (x^1 \vee x^2, y^2).$$

Now, since  $f$  is superadditive, and from the optimality of  $x^1$ ,  
 $x^2$

$$0 \geq f(x^1 \wedge x^2, y^1) - f(x^1, y^1) \geq f(x^2, y^2) - f(x^1 \vee x^2, y^2) \geq 0.$$

Thus

$$f(x^1 \wedge x^2, y^1) = f(x^1, y^1) = \max_{x \in S_{y^1}^*} f(x, y^1)$$

implying  $x^1 \wedge x^2 \in S_{y^1}^*$ . Similarly  $x^1 \vee x^2 \in S_{y^2}^*$ . ■

As a consequence of this theorem and Lemma A10 we have immediately that if  $y^1 \leq y^2$  and both  $S_{y^1}^*$  and  $S_{y^2}^*$  have greatest (least) elements  $\bar{x}^1$  and  $\bar{x}^2$  ( $\underline{x}^1$  and  $\underline{x}^2$ ) respectively, then  $\bar{x}^1 \leq \bar{x}^2$  ( $\underline{x}^1 \leq \underline{x}^2$ ). Furthermore, even if such elements do not exist we can find ordered solutions, since  $x^1 \wedge x^2 \leq x^1 \vee x^2$  and  $x^1 \wedge x^2 \in S_{y^1}^*$   $x^1 \vee x^2 \in S_{y^2}^*$ .

Corollary A12:

Suppose  $S_y$  is a compact sublattice for all  $y$  and  $f(x,y)$  is continuous in  $x$  for all  $y \in \Pi_y S$ . Then  $S_y^*$  is a nonempty compact sublattice for all  $y$ . They have greatest and least elements which are isotonic on  $\Pi_y S$ .

Remark

It is clear that if our problem is one of minimization, we can consider the negative of the function. Equivalently we may define subadditive functions in the obvious manner, and results similar to the above will hold.

To show that  $x(y)$  is decreasing in  $y$ , we can proceed to redefine our problem in terms of  $v = -y$  and show  $x(v)$  is increasing in  $v$ . Thus endless variations of this theme are possible, but they can all be trivially reduced to the case discussed in this appendix.

END